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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 17

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, KINGSWAY, W.C.;
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FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1918

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RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1917–JUNE, 1918.

Thursday, November 1st, 1917.

ANNUAL GENERAL MEETING.

Prof. H. M. MACDONALD, President, and subsequently Prof. E. W. HOBSON, Vice-President, in the Chair.

Present also twelve members and a visitor.

The President announced that the Council had decided to change the hour of meeting from 5.30 to 5: and that, as Prof. W. H. Young was unable to come to England to receive the De Morgan medal in person, it had been arranged that the medal should be transmitted to Sir Adolphus Ward, Master of Peterhouse, to be kept until Prof. Young should be able to come to England to receive the medal.

The President also referred to the deaths of Mr. J. Chevallier and of Baron Kikuchi: both had been members of the Society for over five and twenty years.

The Treasurer presented his Report. On the motion of Mr. J. E. Littlewood, seconded by Mr. L. J. Mordell, the Report was received.

Lt.-Col. A. Cunningham was re-appointed Auditor.

The Council and Officers for the ensuing Session were elected. The list is as follows:—President, Prof. H. M. Macdonald; Vice-Presidents, Prof. H. Hilton, Prof. E. W. Hobson, Prof. Sir J. Larmor; Treasurer, Dr. A. E. Western; Secretaries, Dr. T. J. I'A. Bromwich and Mr. G. H. Hardy; other members of the Council, Prof. W. Burnside, Dr. S. Chapman, Mr. A. L. Dixon, Miss H. P. Hudson, Mr. A. E. Jolliffe, Mr. J. E. Littlewood, Prof. A. E. H. Love, Major P. A. MacMahon, Prof. J. W. Nicholson.

Prof. M. J. M. Hill read a paper "On the Continuation of the Hypergeometric Series."*

Mr. G. H. Hardy communicated a paper by Prof. W. H. Young, "On Restricted Fourier Series and the Convergence of Power Series."*

In the absence of the authors, the following papers were communicated, by title, from the Chair :—

*Tetrahedra in Relation to Spheres and Quadrics Mr. J. H. Grace.

*On Invariants and Covariants of Linear Homogeneous Differential Equations: Prof. E. B. Stouffer.

The Simultaneous System of Two Quaternary Quadratic Forms: Mr. H. W. Turnbull.

ABSTRACTS.

On the Continuation of the Hypergeometric Series

Prof. M. J. M. HILL.

The idea of the continuation of a series is algebraic in origin. Thus, if $P(z)$ be any series, and it be expanded in powers of $z-z_1$, so that another series $P(z|z_1)$ is obtained; and if this be subsequently expanded in powers of $z-z_2$, leading to another series $P(z|z_1|z_2)$, and so on; then each step is algebraic.

Theoretically then it might be supposed that it would be easy to obtain such expansions step by step, but as a matter of fact the labour involved is very great, and it is usual to obtain the continuations either by contour integration or from known results as to the solution of the differential equations satisfied by the series. It would seem, however, to be of interest to obtain the continuations of a few important series by purely algebraic processes.

This was effected by the writer of this paper, in 1902, in the case of the binomial and logarithmic series, each of which has one singular point; and in the case of the series for arc sin x and arc tan x, each of which has two singular points (*Proc. London Math. Soc.*, Ser. 1, Vol. xxxv, pp. 388–416). The object of this paper is to apply the method to the case next in

* Printed in this volume.

order of complexity, namely, to the hypergeometric series, which has three singular points.

It is sufficient to prove the relation between the fundamental solutions of the differential equation satisfied by the two hypergeometric series corresponding to the singularity $x = 0$, and one of those corresponding to the singularity $x = 1$, since from this relation the others of a similar kind can be deduced by the aid of Euler's identities and appropriate transformations of the variable and the parameters.

To simplify the proof it is here supposed that the parameters are real, and that the six fundamental solutions corresponding to the three singular points of the series are finite at the singular points in their respective domains. This makes the parameters α, β, γ and also $\gamma-\alpha-\beta$ positive and less than unity.

As the series $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ cannot be expanded in a convergent series of powers of x, its first $n+1$ terms are taken and rearranged in powers of x. The coefficient of x^r is then transformed, by means of a relation given by the writer (*Proc. London Math. Soc.*, Ser. 2, Vol. 5, pp. 335–338), into two parts. The first part leads directly to the coefficient of x^r in $F(\alpha, \beta, \gamma, x)$ in the required relation between the fundamental solutions.

The second part is an infinite series, the first $n+1$ terms of which contribute to the coefficient of $x^{1-\gamma+k}$ $(0 \leqslant n \leqslant k)$ in

$$x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x),$$

in the relation between the fundamental solutions; but the way in which the coefficients are built up was very difficult to discover, and the work is intricate.

The remaining terms of the infinite series, multiplied by x^r, and the result summed from $r = 0$ to n, are shown to be less than a certain finite multiple of $n^{-(\gamma-\alpha-\beta)}$, which tends to zero as n tends to infinity, since by hypothesis $\gamma-\alpha-\beta > 0$.

Restricted Fourier Series and the Convergence of Power Series

Prof. W. H. YOUNG.

In the present communication a new chapter in the theory of trigonometric series is initiated. The key-note is struck in the following theorem:—

If

$$\sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

is a trigonometrical series (i) *for which* $a_n \to 0$, $b_n \to 0$ $(n \to \infty)$, *and* (ii) *whose integrated series converges in a sub-interval* (α, β) *of the interval of periodicity to an integral* $F(x) = \int f(x)\,dx$, *then the upper and lower functions of the original series at any point* x *of* (α, β) *depend only on the properties of* $f(x)$ *in a neighbourhood of* x *as small as we please.*

In consequence of this theorem, all the convergence properties of Fourier series can be extended to these trigonometrical series, provided we restrict our attention to the interval, or intervals, (α, β).

Moreover the fact that the function

$$Q_n(x) = \frac{d^2}{dx^2} \int_e^{\pi} \left\{ \int_{x-u}^{x+u} F(t)\,dt \right\} \sin nu \cot \tfrac{1}{2}u\,du$$

not only tends to zero when $n \to \infty$ (on which depends the proof of the theorem), but approaches zero uniformly, enables us to include among the convergence properties also uniform convergence properties, and therefore to apply to our series integration theorems known to hold for Fourier series, provided we restrict ourselves to the interval (α, β).

These considerations, among others, have led the author to adopt for trigonometrical series having the properties mentioned in the above enunciation, the term *restricted Fourier series*, or R. F. series. This term was already used by him in a somewhat different sense, which, however, *in the case of convergent series*, is included in the present definition. The series formerly called *restricted Fourier series* satisfied (ii) but not necessarily (i), while on the other hand the integrated series was assumed to be a Fourier series. It was shewn that such series possess all the Cesàro summation properties of Fourier series, but not the ordinary convergence properties.

With the new definition, however, nearly the whole theory of Fourier series can be applied to R. F. series, provided we restrict ourselves to the interval, or intervals (α, β). In particular the three tests for convergence, Dirichlet's, de la Vallée Poussin's, and the author's, may be at once applied to R. F. Series: the function $f(x)$ associated with the series under consideration being defined almost everywhere in (α, β) by the relation

$$f(x) = \frac{dF}{dx}.$$

Again, for example, the R. F. series of a continuous function $f(x)$ will

converge in the Cesàro manner to $f(x)$ at every point x of (α, β) ; more generally, the first derived series of the R. F. series of

$$F(x) = \int f(x)\, dx,$$

where $f(x)$ is continuous, will converge $(C1)$ to $f(x)$ at every point x of (α, β). Also such theorems as those given in the author's paper on "The Integration of Fourier Series" (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 449–462) apply at once to R. F. series.

It is not merely in the regions of convergence and term-by-term integration that the analogy between Fourier series and R. F. series is complete. The second condition in the definition of the latter is in strict accordance with the necessary and sufficient condition that a series should have the Fourier form, given by the author in a earlier communication: the additional property $a_n \to 0$, $b_n \to 0$, is fulfilled of itself in the case of F. series; thus F. series are included in the class of R. F. series. It is easily seen that the chain of theorems based on Riemann's work, due to Lebesgue, Fejér, de la Vallée Poussin, and the author, on certain conditions under which a trigonometric series may be asserted to be a Fourier series, have their place in the theory of R. F. series.

Nor do these theorems exhaust the list of properties common to the whole class of trigonometric series considered and already known for the sub-class of Fourier series. Only those which require the Fourier expressions for the coefficients form an exception.

The second condition in the definition of R. F. series may be put in an alternative form which connects the subject at once with the point of view of Riemann.

We may replace (ii) by the following :—

(ii*a*) *the continuous function* $G(x)$, *to which the second integrated series* $-\sum_{n=1}^{\infty} n^{-2}\{a_n \cos nx + b_n \sin nx\}$ *necessarily converges in virtue of* (i), *is the integral of an integral in* (α, β).

It is easily shewn that the two definitions are identical, and that, almost everywhere in (α, β),

$$f(x) = \frac{d^2 G}{dx^2}.$$

The second integrated series is *necessarily* an F. series. Thus we have two types of R. F. series, firstly, those obtained by term-by-term differentiation of F. series, and secondly those obtained by repeated differentiation, twice over, from F. series.

The results obtained by the author have important applications. In particular they lead, when interpreted in the language of the complex variable, to a notable extension of our knowledge of the behaviour of power series on their circle of convergence. The main theorem shews, for example, that *the uniform convergence of a power series* $\sum_{n=1}^{\infty} c_n z^n$, *where* $c_n \to 0$ $(n \to \infty)$, *can be asserted with confidence if* (i) *the sum function has a unique limiting value at the point* $z = e^{i\theta}$ *in question, and at the points of the circumference in its neighbourhood; and* (ii) *the function of* θ *formed by these limiting values on the circumference is a continuous function of bounded variation.*

This theorem includes as a very special case Fatou's important convergence theorem. Extensions of that theorem also appear as corollaries from the new theory.

In fact, as will be gathered from what precedes, corresponding to the theorems relating to trigonometric series for which a_n and b_n tend towards zero, we have theorems relating to trigonometric series for which this condition is replaced by $n^{-k}a_n \to 0$, $n^{-k}b_n \to 0$, for the various integral values of k and for fractional values. In this connexion the reader of a former paper by the author will realise at once the extension involved corresponding to that in the ordinary theory in which derived series of Fourier series of functions of bounded variation take the place of Fourier series.

Thursday, December 6th, 1917.

Prof. H. HILTON, Vice-President, in the Chair.

Present also fourteen members.

The following were elected members of the Society:—Sir F. Dyson, Messrs. H. D. Anthony and S. Ramanujan.

The Auditor (Lt.-Col. Cunningham) presented his report.

Col. R. L. Hippisley read a paper "A New Method of Describing a Three-Bar Curve."

Lt.-Col. Cunningham and Dr. Western communicated a paper by Mr. O. Hoppe, "Proof of the Primality of $N = \frac{1}{9}(10^{19}-1)$."*

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* A further paper on this subject by Mr. O. Hoppe was communicated on February 14th, 1918; and an Abstract containing the results of both papers will be found in the *Records* of that meeting.

Messrs. Hardy and Littlewood read a paper "Abel's Theorem and its converse."

In the absence of the authors, the following papers were communicated by title from the Chair:—

*On the Curves which lie on the Quartic Surface in Space of Four Dimensions, and the Corresponding Curves on the Cubic Surface and the Quartic with a Double Conic: Mr. C. V. Hanumanta Rao.

(1) On the Connexion between Legendre Series and Fourier Series, (2) On Series of Bessel Functions: Prof. W. H. Young.

ABSTRACTS.

On the Connexion between Legendre Series and Fourier Series

Prof. W. H. YOUNG.

None of the sets of *sufficient* conditions hitherto stated for the convergence of Legendre series

$$\Sigma a_n P_n(x), \tag{1}$$

where

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)\, dx, \tag{2}$$

have been taken to include explicitly the *necessary* condition that the typical term of the series should tend towards zero. The author has already shewn the utility of introducing this condition explicitly in the theory of trigonometrical series of the type which he has called Ordinary Restricted Fourier Series or R.F. series.† These series behave in their interval or intervals of restriction as if they were Fourier series. In the present communication the theory of R.F. series is utilised to prove that *the Legendre series of an absolutely integrable function $f(x)$ behaves in respect of convergence, divergence or oscillation, uniform or non-uniform, at a point or in an interval, exactly like the Fourier series of $f(\cos\theta)$, where $x = \cos\theta$, provided the individual terms of the series tend to zero, that is, provided*

$$n^{-\frac{1}{2}} a_n \to 0 \quad (n \to \infty). \tag{3}$$

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* Printed in this volume.

† See *Records of Proceedings at Meetings*, November 1st, 1917.

The sums of the two series are in fact the same, and the difference of corresponding partial summations tends uniformly to zero. Moreover, the theorem remains true if the series considered be not a Legendre series, but only a series of the form (1), *satisfying* (3) [*but not* (2)], *provided that in some closed interval in* $(-1 < x < 1)$, *containing the point or points considered, the integrated series converges to an integral.*

It is pointed out in the paper that the results arrived at, which include and generalise all known theorems on the convergence of Legendre series, have their analogues in the case of other series of functions, the method employed being perfectly general. In fact, the theory of series of functions connected with the differential equation of the second order, possessing one or more singular points, is found to have the same relation to Restricted Fourier Series as that of series of the Sturm-Liouville type to ordinary Fourier Series.

The author also points out that, if Cesàro convergence be considered instead of ordinary convergence, corresponding results hold, *mutatis mutandis*. Thus, for example, in the case of Legendre series, the Cesàro convergence properties are the same as those of Fourier series, the condition (3) being no longer necessary when the index of convergence is unity, or higher.

On Series of Bessel Functions

Prof. W. H. YOUNG.

In the present communication the principles, already applied by the author to the simplification and extension of the theory of series of Legendre's polynomials, are used to bring into line series of Bessel functions. It is shewn that, *if we assume for a Bessel series*

$$\sum_{r=0}^{\infty} A_r J_m(k_r x), \tag{1}$$

where

$$A_r = C_r \int_0^1 J_m(k_r x) x f(x) dx, \tag{2}$$

the condition, necessary for convergence,

$$A_r J_m(k_r x) \to 0 \quad (r \to \infty), \tag{3}$$

the Bessel series behaves in respect of convergence, divergence, oscillation, uniform or otherwise, precisely like the Fourier series of a function equal

to $f(z)$ in a suitable neighbourhood of the point or points considered, provided those points are internal to the completely open interval (0, 1).

The above theorem is then extended so as to apply to *all series of Bessel functions of the form* (1), *which satisfy* (3), *but not necessarily* (2), *provided that, in the interval* (α, β) *under consideration* $(0 < \alpha < \beta < 1)$, *the sum of the second integrated series is the integral of an integral.*

It is, of course, evident from (3) that this second integrated series converges uniformly, and in an early paragraph of the paper it is established —and the result used in the proof of the theorem as first stated—that the sum is the integral of an integral: thus the second statement of the theorem includes the first as a particular case. It then follows from the theorem, equally in either case, that *the first integrated series converges in* (0, 1) *to an integral.* The function $f(x)$ associated with the Bessel series, or the series of Bessel functions, is then the second differential coefficient of the former integral, and the first differential coefficient of the latter.

The inclusion of series of Bessel functions other than Bessel series not only largely increases the range of series which have become manageable, but enables us to treat satisfactorily a question of fundamental importance in the applications. It has been, in fact, usual to assume tacitly that, if the individual term satisfies a certain differential equation, the function represented by the sum of an infinite number of such terms will also satisfy it. It is now shewn that this conclusion is legitimate, if the function $f(x)$ have in the interval (α, β), in which the equation has to be satisfied, a differential coefficient of suitable order $\phi(x)$, such that the Fourier series of a function, equal in this interval to $\phi(x)$, and having any convenient values elsewhere, converges in the interval; it being presupposed that the individual terms of the series of Bessel functions got by differentiating our Bessel series term-by-term tend to zero. These conditions are, for instance, satisfied, in the case when the differential equation is of the first order, if $f(x)$ is an integral in (0, 1), and the differential coefficient $f'(x)$ has bounded variation in the interval (α, β). Less special forms of this condition are discussed in the paper.

The treatment differs materially in two particulars from that in the paper on Legendre's polynomials. First, the complex variable is introduced to obtain the expansion of x^m in a uniformly convergent Bessel series, an expansion used in various connexions, and, in particular, employed to prove the correlative in our theory of the theorem of Lerch, in virtue of which *two functions which have the same Bessel series agree almost everywhere.* Secondly, except in the special case when m is rational—a case treated preliminarily so as to expose the main features of

the reasoning—the theory of Restricted Fourier Series is insufficient, and a digression is made on the theory of trigonometrical series of the non-harmonic type

$$\Sigma \{a_n \cos(k+n)x + b_n \sin(k+n)x\}.$$

It is shewn that these series have properties analogous to those possessed by ordinary trigonometrical series, equally whether k be rational or irrational.

As regards the behaviour of the series at the end-points of the interval (0, 1), it is briefly indicated that the same method of reasoning, coupled with considerations to be exposed more fully in a subsequent communication, enables us to extend the main result of the paper to the point $x = 1$, at which the trigonometric approximation continues to hold. For the convenience of readers, and for the purpose of comparison, an appendix is added, containing a new and independent proof, on the old lines, of the known fact that the series converges to $f(1-0)$, provided $\sqrt{x}\,f(x)$ is summable and $f(x)$ has bounded variation at the point 1. This is, however, as will be seen, only a special, if important, case of the general theorem indicated.

On the Fourier Series of a Bounded Function

Mr. G. H. HARDY and Mr. J. E. LITTLEWOOD.

Our paper "Abel's Theorem and its converse" contains a statement, but no proof, of the following theorem :—

The Fourier series of a summable function $f(x)$, bounded in the neighbourhood of the particular value of x considered, is either summable (C, k) for ALL *positive values of k (however small) or summable for* NO *value of k (however large). The necessary and sufficient condition for summability is that*

$$\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt \to S \tag{1}$$

when $h \to 0$; and S is then the sum of the series.

As this theorem contains the complete solution of the problem of Cesàro summability for the Fourier series of a bounded function, it seems worth while to indicate the general lines of a proof.

We use unpublished generalisations of three known theorems. In the first place there is a well known theorem (due to M. Riesz and Chapman) that if $f(x)$ is *continuous* then the series is *summable* (C, k) for every posi-

tive k. Substantially the same argument shows that if $f(x)$ is *bounded* then the series is *bounded* (C, k) for every positive k.

In the second place, we have proved* that if a series is bounded (C, r) and summable (C, s), where r and s are integers and $r < s$, then it is summable (C, k) for every integral value of k between r and s. Dr. Riesz afterwards proved that this result remains true when the restriction that r, s and k are integers is removed. This theorem, and others of a similar character, have since been generalised widely by Mr. K. A. Rau, in a dissertation which obtained the Smith's Prize for 1918 in the University of Cambridge; but neither Dr. Riesz's or Mr. Rau's work has yet been published.

It is evident that the first part of our theorem follows at once from a combination of these results.

We have next to show that (1) is necessary and sufficient for summability. That it is *sufficient* follows from a known theorem of Lebesgue,† which asserts that (1) implies summability with $k = 2$. That it is *necessary* may be proved as follows. Fejér has shown‡ that, if the series is summable $(C, 1)$ to sum S, then

$$\frac{\phi(h)}{16h^4} = \frac{\Delta^4_{2h} F_4(x)}{16h^4}$$

$$= \frac{F_4(x+4h) - 4F_4(x+2h) + 6F_4(x) - 4F_4(x-2h) + F_4(x-4h)}{16h^4} \to S$$

when $h \to 0$, $F_4(x)$ being the fourth iterated integral of $f(x)$. Taking, as we may do without loss of generality, $S = 0$, we have

$$\phi(h) = o(h^4).$$

Also
$$\phi'''(h) = 64\{\chi(h) - \tfrac{1}{2}\chi(\tfrac{1}{2}h)\},$$

where
$$\chi(h) = F_1(x+4h) - F_1(x-4h)$$

---

* "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 411–478. The particular result referred to is not stated as an explicit theorem anywhere in our paper; but it is an immediate deduction from Theorem 3. It is also virtually included in Theorem 19. But the latter theorem involves a parameter β which we assume to be positive; and here β is zero, so that some restatement of the argument is required.

The work of Dr. Riesz to which we refer is contained in a letter addressed to one of us on November 28th, 1912. In this letter Dr. Riesz generalises our Theorem 3; and the theorem required here is an immediate deduction from his generalisation.

† "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, Vol. 61, pp. 251–280 (pp. 278–279).

‡ "Untersuchungen über Fourierschen Reihen", *Math. Annalen*, Vol. 58, pp. 51–69 (p. 69).

is the integral of a bounded function. From this it is easily deduced, by arguments similar to those of Section II of our paper already referred to, that $\phi'''(h) = o(h)$*, and hence that $\chi(h) = o(h)$; which completes the proof of our theorem.

Thursday, January 17th, 1918.

Prof. H. HILTON, Vice-President, and subsequently Dr. A. E. WESTERN, Hon. Treasurer, in the Chair.

Present also eight further members and a visitor.

Mr. G. H. Hardy read a paper "Additional Note on Dirichlet's Divisor Problem," and communicated various definite integrals recently evaluated by Mr. S. Ramanujan.

Prof. Hilton, having called Dr. Western to the Chair, communicated a paper written by himself and Miss Tuck, "Plane Quartic Curves with a Tac-Node."

In the absence of the authors the following papers were communicated by title from the Chair :—

On a Method for Studying any Convergent Series: Major P. A. MacMahon.

†Note on Diophantine Approximation: Mr. J. H. Grace.

†A Note on a Theorem of Mr. Hardy's: Mr. K. Ananda Rau.

Super-Normal Curves: Mr. C. H. Forsyth.

ABSTPACTS.

On Plane Quartic Curves with a Tac-Node

Prof. HAROLD HILTON and Miss D. S. TUCK.

The equation of such a quartic may be put in the form

$$(yz+x^2)^2 = (1-m)(x+ay)(x+by)(x+cy)(x+dy);$$

* This is the third "unpublished generalisation" to which we referred above. For generalisations of some of the older theorems in this direction see Landau, "Einige Ungleichungen für zweimal differentiierbare Funktionen", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 43–49.

† Printed in this volume.

or, if the curvatures at the tac-node are equal and opposite, in the form

$$My^2z^2+(x+ay)(x+by)(x+cy)(x+dy) = 0.$$

In the former case the bitangents can be grouped into three pairs, of which one pair is

$$2z = \{(a+b+c+d)x+(ab+cd)y\} \pm \sqrt{m}\{(a+b-c-d)x+(ab-cd)y\};$$

and similarly in the latter case.

The tangents from the tac-node are $x+ay = 0$, &c.

Many results can be obtained by means of these equations: the theorems which follow are examples. (1) The pairs of bitangents meet the tangent at the tac-node in an involution. (2) The points of contact of a pair of bitangents lie on a conic touching the quartic at the tac-node. (3) The points of contact of two pairs of bitangents lie on a conic. (4) Two conics can be drawn through the points of contact of a given bitangent, touching the quartic at the tacnode, and passing through two of the points of contact of tangents from the tac-node.

If the quartic has another double point, $c = d = 0$; or $b = c = d = 0$, if the other double point is a cusp. As before, we have a series of theorems. (1) The points of contact of either bitangent, and its intersections with the tangents from the node, form an involution with a double point on the tangent at the tac-node. (2) The two bitangents meet on the line joining the points of contact of the two tangents from the tac-node. (3) A conic touches the tangents from the node and touches the quartic at its points of contact with the tangents from the tac-node. And so on.

Tacnodal quartics of deficiency 1 can be classified into eight different types according as $m > 1$, $1 > m > 0$, $0 > m$; and according as four, two, or none of the tangents from the tac-node are real. The types of unicursal tacnodal quartic can be deduced as limiting cases.

The paper will appear in full in the *Messenger of Mathematics*.

Note on Diophantine Approximation

Mr. J. H. GRACE.

Minkowski has shewn that if a is an irrational number, and b not an integer, there is an infinite number of pairs of integers x, y such that

$$|ay-x-b| < \frac{1}{4|y|}.$$

It is here proved that this result is final, *i.e.* that, if the factor $\frac{1}{4}$ is replaced by a smaller one, it is possible to choose a and b so that the above inequality does not hold for an infinite number of values of x and y. The case used is that in which $b = \frac{1}{2}$ and a is a continued fraction whose first quotient is odd and whose subsequent quotients are all even and increase steadily.

Some Definite Integrals

Mr. S. RAMANUJAN.

Typical formulæ are:—

$$(1) \qquad \int_{-\infty}^{\infty} \frac{e^{nix}\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\, e^{\frac{1}{2}in(\beta-\alpha)i} \quad (or\ 0),$$

$$(2) \qquad \int_{-x}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\, e^{nix}\,dx = \pm\frac{2\pi i(2\sin\frac{1}{2}N)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\, e^{-n\alpha i+\frac{1}{2}(\pi-N)(\beta-\alpha-1)i} \quad (or\ 0),$$

$$(3) \qquad \int_{-\infty}^{\infty} \Gamma(\alpha+x)\,\Gamma(\beta-x)\,e^{nix}\,dx$$

$$= \frac{2\pi i\Gamma(\alpha+\beta)}{(2\sin\frac{1}{2}N)^{\alpha+\beta}}\, e^{\frac{1}{2}in(\beta-\alpha)i}\left[\epsilon_n(\beta)\,e^{k\pi(\alpha+\beta)i} - \epsilon_n(-\alpha)\,e^{-k\pi(\alpha+\beta)i}\right].$$

Here n is real, $n = 2k\pi+N$ $(0 \leqslant N < 2\pi)$ in (2), and $n = (2k-1)\pi+N$ $(0 \leqslant N < 2\pi)$ in (3). In (1) the zero value is to be taken if $|n| \geqslant \pi$, the non-zero value otherwise. In (2) α must be complex: the zero value is to be taken if n and $\mathfrak{I}(\alpha)$ have the same sign, the positive sign if $n \geqslant 0$ and $\mathfrak{I}(\alpha) < 0$, and the negative sign if $n \leqslant 0$ and $\mathfrak{I}(\alpha) > 0$. In (3) α and β must both be complex; and $\epsilon_n(\zeta)$ is 0, 1, or -1 according as (i) $\pi-n$ and $\mathfrak{I}(\zeta)$ have the same sign, (ii) $n \leqslant \pi$ and $\mathfrak{I}(\zeta) < 0$, (iii) $n \geqslant \pi$ and $\mathfrak{I}(\zeta) > 0$.

The convergence conditions are, in general, (1) $\mathfrak{R}(\alpha+\beta) > 1$, (2) $\mathfrak{R}(\alpha-\beta) < 0$, (3) $\mathfrak{R}(\alpha+\beta) < 1$. But there are certain special cases in which a more stringent condition is required.

A formula of a different character, deduced from (1), is

$$\int_{-\infty}^{\infty} \frac{J_{\alpha+x}(\lambda)}{\lambda^{\alpha+x}}\,\frac{J_{\beta-x}(\mu)}{\mu^{\beta-x}}\, e^{nix}\,dx = \left(\frac{2\cos\frac{1}{2}n}{\Omega}\right)^{\frac{1}{2}(\alpha+\beta)} e^{\frac{1}{2}in(\beta-\alpha)i}\, J_{\alpha+\beta}\left\{\sqrt{(2\Omega\cos\tfrac{1}{2}n)}\right\}$$

(*or* 0). Here $\Omega = \lambda^2 e^{\frac{1}{2}ni} + \mu^2 e^{-\frac{1}{2}ni}$;

the zero value is to be taken if $|n| \geqslant \pi$, the non-zero value otherwise;

and the condition of convergence is, in general, that

$$\Re(\alpha+\beta) > -1.$$

The formulæ include a large number of interesting special cases, such as

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)},$$

$$\int_0^{\infty} \frac{\sin \pi x\, dx}{x(x^2-1^2)(x^2-2^2)\ldots(x^2-k^2)} = (-1)^k \frac{2^{2k-1}\pi}{2k!},$$

$$\int_{-\infty}^{\infty} J_{\alpha+x}(\lambda)\, J_{\beta-x}(\lambda)\, dx = J_{\alpha+\beta}(2\lambda).$$

The formula

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)},$$

may also be mentioned: it holds, in general, if

$$\Re(\alpha+\beta+\gamma+\delta) > 3.$$

A fuller account of these formulæ will be published in the *Quarterly Journal of Mathematics.*

Thursday, February 14th, 1918.

Prof. H. HILTON, Vice-President, in the Chair.

Present also fourteen members and a visitor.

Mr. G. H. Hardy, as Secretary, announced that at the beginning of the current Session (November 1917), the number of members on the list was 296 as against 299 at the beginning of the previous Session; 3 new members were elected, and 6 members died during the previous Session.

Lt.-Col. Cunningham and Dr. Western gave an account of a further investigation by Mr. O. Hoppe* on "The Primality of $\frac{1}{9}(10^{19}-1)$."

Lt.-Col. Cunningham also gave a short account of the history of the number $\frac{1}{9}(10^{17}-1)$.

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* See *Records of Proceedings*, December 6th, 1917.

Mr. L. J. Mordell read a paper "On a Statement by Fermat."

In the absence of the authors, the following papers were communicated by title from the Chair:—

Note on Functional Equations which are Limiting Forms of Integral Equations: Prof. A. C. Dixon.

The Singularities of Trochoidal Curves: Prof. D. M. Y. Sommerville.

ABSTRACTS.

The Primality of $\frac{1}{9}(10^{19}-1)$

Mr. OSCAR HOPPE.

*Mr. Oscar Hoppe's papers deal with the number $N = \frac{1}{9}(10^{19}-1)$, and contain a summary of the results of his calculations. He first employed the process for factorising large numbers contained in Prof. F. N. Cole's paper (*Bulletin American Math. Soc.*, Ser. 2, Vol. 10, p. 134). He thus proved that p is a quadratic residue of every factor of N, where p represents any prime, less than 761, of which N is a quadratic residue, those of the form $4n+3$ being taken with a minus sign.

He then searched for possible prime factors of N up to a limit about 23.10^6, as follows. From the properties of such factors to moduli 19, 8, 3, 5, 7, 11, 13, every prime factor must be congruent to one of 360 residues (mod 2282280). All the possible factors of these forms were excluded by using the tests of the quadratic character (mods 17, 19, ..., 127).

Finally, he searched for solutions of $N = x^2-y^2$, excluding possible values of x in a similar manner. The result was that no factors were discovered, and that therefore N is a prime.

He also found that $3^{N-1} \equiv 1 \pmod{N}$, which does not furnish a proof that N is prime, but confirms the accuracy of the other calculations.

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* This Abstract (prepared by Dr. Western) gives a summary of the results obtained in two papers communicated at the meetings on December 6th, 1917, and February 14th, 1918.

Note on Functional Equations which are Limiting Forms of Integral Equations

Prof. A. C. DIXON.

I believe that some progress could be made in the theory of integral equations by the study of such equations as

$$\phi(x)+\phi(X_1)+\phi(X_2)+\ldots+\phi(X_m)+\int_a^b K(x, y)\,\phi(y)\,dy = f(x), \qquad (1)$$

where ϕ is the unknown function, and X_1, X_2, ..., X_m, $f(x)$ are known functions of x. Since $\phi(X)$ can be expressed in many ways as

$$\operatorname*{Lt}_{n\to\infty} \int_a^b g(y, X, n)\,\phi(y)\,dy,$$

these equations are limiting forms of integral equations.

A simple case which can be fully worked out is that of the equation

$$\phi(x)-\theta\,\phi(x+a) = f(x), \qquad (3)$$

where ϕ, f are supposed to have the period 2π and θ is a parameter.

If $f(x)$ is bounded, the solution, when $|\theta| < 1$, is given by

$$\phi(x) = f(x)+\theta f(x+a)+\theta^2 f(x+2a)+\ldots,$$

and when $|\theta| > 1$ by

$$\phi(x) = -\frac{1}{\theta} f(x-a)-\frac{1}{\theta^2} f(x-2a)-\frac{1}{\theta^3} f(x-3a)-\ldots,$$

the singular values of θ being all found on the circle $|\theta| = 1$.

Any integral power of $e^{\iota a}$ is a singular value of θ, and the special function corresponding to $e^{m\iota a}$ is $e^{-m\iota x}$.

Thus, if $f(x)$ can be expanded as a Fourier series

$$\Sigma a_m e^{\iota m x},$$

the expansion of $\phi(x)$ is $$\Sigma \frac{a_m e^{\iota m x}}{1-\theta e^{\iota m a}}.$$

The associated equation, corresponding to the change of rows into columns in a determinant, can also be formed.

If $\psi(x)$ is any function with the same period,

$$\int_0^{2\pi} \psi(x)\{\phi(x)-\theta\phi(x+a)\}\,dx = \int_0^{2\pi} \{\psi(x)-\theta\psi(x-a)\}\,\phi(x)\,dx,$$

and the associated equation is

$$\psi(x)-\theta\psi(x-a) = f(x). \tag{4}$$

Here again e^{mia} is a singular value of θ, the associated function being now e^{mix}. If the functions associated with e^{mia} are $f_m(x)$, $g_m(x)$ for the equations (3), (4) respectively, we have, in full accordance with the general theory,

$$\int_0^{2\pi} f_m(x)\,g_n(x)\,dx = 0, \quad (m \neq n),$$
$$= 2\pi, \quad (m = n).$$

If $\phi(x)$, the function determined by (3), is treated as a function of θ, it has the circle $|\theta| = 1$ as a singular line or natural boundary, unless a is commensurable with π. The function exists on both sides of the boundary but cannot be continued across it, except for special forms of $f(x)$.

On the other hand, in the theory of the integral equation of the usual type

$$\phi(x)-\theta\int_a^b K(x, y)\,\phi(y)\,dy = f(x), \tag{5}$$

the function $\phi(x)$ has no essential singularity in the θ-plane except at infinity.

Further, $\phi'(x)$ can also be expressed in the form (2), and thus the equation

$$\phi(x)-\theta\phi'(x) = f(x) \tag{6}$$

is also included among the limiting forms of (5); we still suppose f, ϕ to have the period 2π.

The value of the periodic function $\phi(x)$ is here

$$\frac{e^{x/\theta}}{\theta(e^{2\pi/\theta}-1)}\left[e^{2\pi/\theta}\int_0^{2\pi} e^{-t/\theta}f(t)\,dt-(e^{2\pi/\theta}-1)\int_0^{x} e^{-t/\theta}f(t)\,dt\right],$$

and its essential singularity occurs at $\theta = 0$. The other singular values of θ are ∞, $\pm\iota$, $\pm\frac{\iota}{2}$, $\pm\frac{\iota}{3}$, ..., and when $\theta = \frac{\iota}{m}$ the associated function is $e^{x/\theta}$, that is e^{-mix}.

The associated equation is

$$\psi(x)+\theta\psi'(x) = f(x),$$

since when ϕ, ψ are periodic

$$\int_0^{2\pi} \psi(x)\{\phi(x)-\theta\phi'(x)\}\,dx = \int_0^{2\pi} \phi(x)\{\psi(x)+\theta\psi'(x)\}\,dx.$$

Thursday, March 14th, 1918.

Prof. E. W. HOBSON, Vice-President, in the Chair.

Present also sixteen members.

Mr. G. H. Hardy communicated a paper "On the representation of a number as the sum of any number of squares, and in particular of five or seven."

Mr. G. N. Watson made an informal communication on "A Problem in the Theory of Numbers."

The following paper, in the absence of the author, was communicated, by title, from the chair :—

On Non-Harmonic Fourier Series : Prof. W. H. Young.

ABSTRACTS.

*On the representation of a number as the sum of any number of squares, and in particular of five or seven**

Mr. G. H. HARDY.

The central formulæ in the theory of the representation of numbers by sums of five or seven squares are

$$r_5(n) = \frac{Bn\sqrt{n}}{\pi^2}\,\Sigma\left(\frac{n}{m}\right)\frac{1}{m^2}, \tag{1}$$

and

$$r_7(n) = \frac{Cn^2\sqrt{n}}{\pi^3}\,\Sigma\left(\frac{-n}{m}\right)\frac{1}{m^3}.\dagger \tag{2}$$

Here $r_5(n)$ and $r_7(n)$ are the coefficients of q^n in ϑ^5 and ϑ^7, where

$$\vartheta = 1+2q+2q^4+2q^9+\ldots;$$

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* A somewhat fuller account of this paper has been published in the *Proceedings of the National Academy of Sciences* (Washington), Vol. 4, 1918, pp. 189–193.

† Bachmann, *Zahlentheorie*, Vol. 4, pp. 621, 655.

n is an odd number not divisible by any square (so that there is no distinction between primitive and imprimitive representations); $\left(\frac{n}{m}\right)$ is the symbol of Legendre and Jacobi; m runs through all odd numbers prime to n; B is 80, 160, 112, or 160, according as n is congruent to 1, 3, 5, or 7 (mod 8); and C is 448, 560, 448, or 592 in similar circumstances. The series on the right-hand side may be summed in finite terms (in terms of the residues and non-residues of n) by well known methods due to Dirichlet and Cauchy.

These formulæ have never been proved except by means of the general theory of arithmetic forms, due to Eisenstein, Smith, and Minkowski. In this paper they are deduced from the theory of the elliptic modular functions; and the theory of the representation of a number as the sum of an odd number of squares is thus brought into line with the similar theory for an even number.

The fundamental formulæ, which embody the whole theory, are as follows:—

$$(3)\quad \mathfrak{Z}^5 = 1+\tfrac{32}{3}\left\{\sum_{1,3,5,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{3}{2}}q^{mk+j} - \sum_{2,4,6,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{\frac{3}{2}}q^{mk+j}\right\},$$

$$(4)\quad \mathfrak{Z}^7 = 1+\tfrac{256}{15}\left\{\sum_{1,3,5,\ldots}\frac{(-1)^{\frac{1}{2}(k-1)}}{k^3}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{5}{2}}q^{mk+j} - \sum_{2,4,6,\ldots}\frac{1}{k^3}\sum_{j'}\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j')^{\frac{5}{2}}q^{mk+j'}\right\}.$$

Here j runs through a complete set of least positive residues (mod k) of $0^2, 1^2, 2^2, \ldots, (k-1)^2$, each such residue being counted as often as it occurs; j' through a corresponding set of residues of $\frac{1}{2}k, \frac{1}{2}k+1^2, \ldots, \frac{1}{2}k+(k-1)^2$; and μk is in either case the multiple of k which must be deducted in order to arrive at such a residue. There is a similar formula for $\mathfrak{Z}^3$, which embodies the theory for *three* squares.

The results were arrived at incidentally during the course of researches, originally undertaken in conjunction with Mr. Ramanujan, with a different aim, viz. that of finding *asymptotic* formulæ for $r_s(n)$ for *all* values of s. The dominant term of the asymptotic formula arrived at was *identified* with $r_s(n)$, when $s = 4$, 6, or 8, by Mr. Ramanujan, in a paper shortly to appear in the *Transactions of the Cambridge Philosophical Society*. My method, which is quite unlike his, provides an alternative proof of his

following:—

$$s_n = \sum_{r=-n}^{n} A_r^{(k)} = \tfrac{1}{2}\{A_n^{(k)} + A_{-n}^{(k)}\}$$

$$+ \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du.$$

The proof of these properties depends on two fundamental lemmas, which in the present theory take the place of the normal properties of the sine and cosine. These are

$$\int_0^{2\pi} \begin{matrix}\sin\\ \cos\end{matrix} (n+k)(2\pi-u) \begin{matrix}\cos\\ \sin\end{matrix} (r+k)\, u\, du \equiv 0 \quad (r \neq n)$$

$$= \pm \pi \sin 2k\pi \quad (r = n),$$

r and n being integers, positive, negative, or zero.

It follows from the theorem of Riemann-Lebesgue that the coefficients in a non-harmonic Fourier series, like those of a harmonic Fourier series, tend to zero. A number of other well-known theorems for harmonic series hold equally for non-harmonic ones, in particular the necessary and sufficient condition that a series should have the Fourier form, and Riemann's three theorems on trigonometrical series, as well as Harnack's theorem giving certain conditions under which the coefficients of a trigonometrical series tend to zero. These generalisations having been established, we come to the principal result of the second part of the paper, the proof of the unicity of the expansion of a function in a non-harmonic cosine or sine series, under the same conditions as for a harmonic series.

The main object of the third part of the paper is to apply the considerations which precede to establish rigidly a statement made by the author in a previous paper, as to the behaviour of a Bessel series at the point $z = 1$. For this purpose a brief account is given of *restricted non-harmonic Fourier series*. In the light of the "continuation" formulæ, this theory is found to apply, not only when the interval of restriction belongs to the completely open interval $(0, 2\pi)$, but even when one of these points is an included end-point of the interval of restriction.

The result is that *the conditions of convergence of a Bessel series, at any point of the half open interval* $(0 < z \leqslant 1)$, *are the same as those of the Fourier series of a function equal to the associated function* $f(z)$ *of the Bessel series in an arbitrary small neighbourhood surrounding the point considered and having any convenient values elsewhere, provided the function be "continued" by reflection in the point* $z = 1$.

Thursday, April 18th, 1918.

Prof. E. W. HOBSON, Vice-President, in the Chair.

Present also eleven other members.

Major MacMahon made an informal communication "On the attraction of a circular lamina." Remarks were made on the communication by Prof. Hobson and Dr. Bromwich.*

Prof. H. Hilton made a communication "On n-poled cassinoids."

ABSTRACT.

On n-poled cassinoids.

Prof. HAROLD HILTON.

The n-poled cassinoid is the curve of degree $2n$,

$$r^{2n} - 2r^n c^n \cos n\theta + c^{2n} = a^{2n},$$

which is the locus of a point P whose distances from the vertices of a regular n-sided polygon with centre O have a constant product. The n tangents at each circular point have $(n+1)$-point contact, and the n tangents from each circular point have n-point contact.

The product of the distances of P from the ordinary foci is proportional to OP^n.

The angle between OP and the normal at P is the algebraic sum of the angles subtended at P by the lines joining O to the singular foci.

The locus of a point P such that

$$PL_1 \cdot PL_2 \ldots PL_n / PM_1 \cdot PM_2 \ldots PM_n$$

is constant, where $L_1 L_2 \ldots L_n$ and $M_1 M_2 \ldots M_n$ are two concentric regular polygons, is an n-poled cassinoid; and so is the locus of a point at which $L_1 M_1$, $L_2 M_2$, ..., $L_n M_n$ subtend angles with a constant algebraic sum.

If two cassinoids with singular (or ordinary) foci at the vertices of two given concentric regular polygons cut at a constant angle, the locus of their intersections is a cassinoid.

The locus of a point whose polar conic with respect to a given cassi-

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* See also *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 100.

noid has constant eccentricity is a cassinoid with ordinary foci at the singular foci of the given cassinoid.

A cassinoid may be considered as the envelope of a circle cutting a fixed circle orthogonally and having its centre on a curve of class n, whose foci are the singular foci of the cassinoid.

The orthogonal trajectories, locus of inflexions, locus of points of contact of tangents from the centre, &c., for cassinoids with given singular or ordinary foci, are curves of the type $r^n = b^n \cos n\theta$ or $r^n \cos n\theta = b^n$.

Thursday, May 9th, 1918.

Prof. H. HILTON, Vice-President, in the Chair.

Present also twelve other members.

Mr. K. Ananda Rau was elected a member of the Society.

Dr. A. E. Western made an informal communication on "The Factorisation of Fermat's Numbers."

Dr. G. N. Watson spoke further on the problem in the theory of numbers, raised by him at the March meeting.

Mr. L. J. Isserlis made an informal communication on "A Definite Integral of importance in the Theory of Statistics."

In the absence of the authors, the following papers were communicated by title from the Chair:—

On the Continued Fractions Associated with the Hypergeometric Equation: Mr. E. Lindsay Ince.

Determinantal Systems of Co-Apolar Triads on a Cubic Curve: Dr. W. P. Milne.

The Electromagnetic Properties of Coils: Dr. A. Young.

ABSTRACTS.

The Factorisation of Fermat's Numbers

Dr. A. E. WESTERN.

Fermat's numbers are $F_n = 2^{2^n}+1$. It is known that F_7 and F_8 are

composite, but no factors of these numbers have been discovered*; it is known also that every factor of F_n is congruent to 1 (mod 2^{n+2}).

Let p be a prime factor of F_n; then

$$p = a^2+b^2,$$

where $b \equiv 0 \pmod 4$. If F_n/p is prime, and

$$F_n/p = a'^2+b'^2,$$

then
$$F_n = (aa' \mp bb')^2+(ab' \pm a'b)^2.$$

One of these representations of F_n must be $1+2^{2^n}$, and we call the other A^2+B^2. And if F_n/p is composite, then among its representations in the form x^2+y^2 there must be one, say $a'^2+b'^2$, which by composition with a^2+b^2 gives F_n in the form $1+2^{2^n}$. Thus we obtain

$$ab'+a'b = 2^{2^{n-1}}, \tag{1}$$

$$ab'-a'b = B,$$

and
$$2ab' = B+2^{2^{n-1}}. \tag{2}$$

It is evident from (1) that the highest power of 2 in b and b' is the same, say 2^x, and from (2) that B is a multiple of 2^{x+1} and of no higher power of 2. It follows that

$$B \equiv 0 \pmod 8.$$

The author suggests that the best method of factorising F_7 or F_8 will be to search for values of A satisfying

$$F_n = A^2+B^2.$$

Since $B \equiv 0 \pmod 8$, $A \equiv \pm (2^{2^{t-1}}-1) \pmod{2^{2^{t+2}}}$, $(t = 3, 4, \ldots)$.

Each solution A, B, gives a factorisation of F_n into two factors, for the greatest common factor of $2^{2^{n-1}}A-B$ and F_n is a factor of F_n other than 1 and F_n.

The Electromagnetic Properties of Coils

Dr. A. YOUNG.

The magnetic potential of a closed circuit, at any point not in the

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 3, p. xxi, and *Bulletin American Math. Soc.*, Vol. 16, p. 1.

circuit, is equal to ωi, where ω is the solid angle subtended by the circuit at the point, and i is the strength of the current. The magnetic potential of a coil can be obtained by summing the expressions ωi for each turn. To find the magnetic induction parallel to the axis, we take the difference between the potentials at two neighbouring points such that the straight line joining them is parallel to the axis. The effect would be the same if we took the difference between the potentials of two coils at the same point, the one coil being obtained from the other by a small displacement parallel to the axis. If the coil be a one-layer coil having n turns to the centimetre (measured parallel to the axis) we see that the magnetic induction parallel to the axis is $n\Omega i$, where Ω is the difference between the solid angles subtended at the point by the top and bottom turns of the coil. If the point be inside the coil, Ω is the solid angle subtended at the point by the coil itself. This gives a very simple method for seeing at a glance the relative magnitudes of the induction parallel to the axis due to a coil at all points in space.

In the case of a rectangle, the solid angle which it subtends at any point may be obtained as the inverse tangent of a certain ratio: this is done. And hence is deduced the magnetic potential in all space of a coil whose section perpendicular to its axis is rectangular. If the coil have n turns to the centimetre, and i be the current strength, this potential is, in general,

$$\phi = \Sigma \left[\xi \tan^{-1} \frac{\xi\eta}{\zeta r} + \tfrac{1}{2}\xi \log \frac{r+\eta}{r-\eta} + \tfrac{1}{2}\eta \log \frac{r+\xi}{r-\xi} \right] ni.$$

Here ξ, η, ζ have the values $a \pm x$, $b \pm y$, $z \pm h$, and $r^2 = \xi^2 + \eta^2 + \zeta^2$; each turn of the coil is a rectangle whose sides are $2a$, $2b$, and the height of the coil is $2h$; the origin is at the middle point of the axis, and the axes of coordinates are parallel to the sides of the rectangles and to the axis of the coil respectively. If, however, the point is in the interior of the coil, the potential is

$$\phi + 2\pi (h - z)\, ni.$$

As the function ϕ can be integrated with respect to both coordinates x and y, the self inductance of the coil can be deduced, and is given. Also the mutual inductance of any two such coils whose sides are parallel is obtained.

It is further shewn how to integrate ϕ over any rectangular area; and hence how to deduce the mutual inductance of any two such coils, whatever their relative positions. The fact that the integrals involve no transcendental functions except the inverse tangent and the logarithm is important from the point of view of numerical calculation.

Thursday, June 13th, 1918.

Prof. E. W. HOBSON, Vice-President, in the Chair.

Present also eighteen other members and four visitors.

Prof. M. J. M. Hill read a paper "An Assumption in the Theory of Singular Solutions of Ordinary Differential Equations of the First Order."

Lt.-Col. A. J. Cunningham communicated two papers, "Quartic and Cubic Residuacity Tables" and "Lucas's Process applied to Composite Mersenne's Numbers." The first paper was written in collaboration with Mr. Th. Gosset.

Dr. A. E. Western read a paper "The Gaussian Period Numbers and the Conditions that 2 should be a Residue of a 16th or 32nd Power."

Mr. T. W. Chaundy read a paper "The Aberrations of a Symmetrical Optical System."

Mr. E. Lindsay Ince read a paper "The Rotation Groups of the Regular Figures in Four or More Divisions."

In the absence of the authors, the following papers were communicated by title from the Chair:—

Chords of Twisted Cubics: Mr. E. K. Wakeford.*

(1) An Analogue in Space of a Case of Poncelet's Porism, (2) A Note on Enumerative Geometry: Mr. J. H. Grace.

ABSTRACTS.

Quartic and Cubic Residuacity Tables

Lt.-Col. ALLAN CUNNINGHAM, R.E., and Mr. TH. GOSSET.

This memoir explains the construction and use of Tables of Criteria of Quartic and Cubic Residuacity, showing whether

$(Q/p)_4 =$, or $\neq \mp 1$, where $p = 4\varpi+1 = a^2+b^2$, *a prime*,

$(Q/p)_3 =$, or $\neq 1$, where $p = 3\varpi+1 = \frac{1}{4}(L^2+27M^2)$, *a prime*,

Q being any integer prime to p. If Q is a product of prime factors q,

$(Q/p)_4$ is shown to depend on the ratios a/b or b/a (mod q).

$(Q/p)_3$ is shown to depend on the ratios L/M or M/L (mod q).

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* This MS. was discovered in Lieut. Wakeford's kit after his death.

Tables are given of the integer values of the ratios a/b, b/a (mod q), and of the ratios L/M, M/L (mod q), for all primes $q \not> 50$, thus giving directly (with slight computation) the criteria of $(Q/p)_4$ and $(Q/p)_3$ for *all* bases Q which are products of prime factors q (each < 50).

These tables are computed as solutions of the congruences

$$\left(\frac{a+bc}{a-bc}\right)^{\frac{1}{4}(q-1)} \equiv c,\ -1,\ -c,\ +1 \pmod{q},$$

$$\left(\frac{L+27M}{L-27M}\right)^{\frac{1}{3}(q-1)} \equiv \rho,\ \rho^2,\ 1 \pmod{q}, \quad [\rho^3-1=0];$$

these being special forms of the Laws of Quartic and Cubic Reciprocity which are convenient for this computation.

On Lucas's Process applied to Composite Mersenne's Numbers

Lt.-Col. ALLAN CUNNINGHAM, R.E.

Ed. Lucas developed a process (see *American Journal of Mathematics*, Vol. 1, 1878, pp. 303–316) for testing the *primality* of Mersenne's Numbers $M_q = 2^q-1$ [q *prime*] by use of the series $v_1, v_2, v_3, \ldots$, defined by

$$v_{n+1} \equiv v_n^2-2 \pmod{M_q};$$

showing that (with a suitable v_1)

(1) M_q is *prime* if $v_n \equiv 0 \pmod{M_q}$ with $n > \frac{1}{2}q$ but $< q$;

(2) $v_n \not\equiv 0 \pmod{M_q}$ if M_q is *composite*, but the series of v_n's is periodic.

In the latter case let $v_{\rho+1}$ be the *first* term that repeats, so that, after say ν steps,

$$v_{\nu+\rho+1} \equiv v_{\rho+1} \pmod{M_q}.$$

This involves $$v_{\nu+\rho}^2 - v_\rho^2 \equiv 0 \pmod{M_q}.$$

The possibility of using this result for finding the factors of a composite M_q is discussed, and the utilisation of the steps already worked (q in number) in an attempt to test the primality of any M_q (if thereby proved composite) is also explained.

The Gaussian Period Numbers, and the Conditions that 2 should be a Residue of a 16-th or a 32-nd Power

Dr. A. E. WESTERN.

It has long been known that there is a set of positive integers associated with the m-termed Gaussian periods $\eta_0, \eta_1, \ldots$ of the p-th roots of 1, where p is a prime and m is a factor of $p-1$. These numbers are the coefficients (h, k) in the equations

$$\eta_0\eta_k = \sum_k (h, k)\,\eta_k \quad \left(h,\ k = 0, 1, \ldots, \frac{1}{m}(p-1)-1\right).$$

The author proposes to call these numbers *the Gaussian numbers of degree* $(p-1)/m$.

We take p to be ml^n+1, where l is a prime; ξ is a primitive l^n-th root of 1, and $\mathfrak{p}$ is one of the prime factors of p in the field of algebraic numbers defined by ξ. And $\{\alpha/\mathfrak{p}\}$ denotes that power of ξ which is congruent to $\alpha^{(p-1)/l^n} \pmod{\mathfrak{p}}$.

Kummer has evaluated $\{(1-\xi^x)/\mathfrak{p}\}$ in terms of the Gaussian numbers of degree l^n.* The author has found expressions for the Gaussian numbers in terms of the coefficients of the reciprocal factors ψ of p in the field of ξ.† Consequently, for any value of l^n, $\{(1-\xi^x)/\mathfrak{p}\}$ can be expressed in terms of the reciprocal factors. And since

$$l = \prod_x (1-\xi^x),$$

where x takes all values prime to l from 1 to l^n-1, $\{l/\mathfrak{p}\}$ can also be thus expressed.

In particular, taking $l = 2$, we have

$$2 = \xi^4(1-\xi)^2(1-\xi^3)^2,$$

when $n = 3$, and $\quad 2 = \prod(1-\xi^{2x+1})^2 \quad (x = 0, 1, \ldots, 2^{n-2}-1)$

when $n > 3$.

As 2 is a square number in these fields, we see that the value of $2^{(p-1)/2^{n+1}} \pmod{p}$ is expressible in terms of the reciprocal factors of p of degree 2^n. The author has in this manner calculated the general value

* *Journal für Math.*, Vol. 44 (1851).

† See *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 16 and 265, for the definition and properties of the reciprocal factors.

of $\{2/\mathfrak{p}\}_{16}$ and $\{2/\mathfrak{p}\}_{32}$,* but it will suffice to mention here the following special cases.

I. When $p \equiv 1 \pmod{16} = a^2+b^2 = c^2+2d^2$,

and $b = 0 \pmod{16}$,

so that $2^{(p-1)/8} \equiv 1 \pmod{p}$,

then $2^{(p-1)/16} \equiv (-1)^{\frac{1}{16}b+\frac{1}{4}d} \pmod{p}$.†

II. When $p \equiv 1 \pmod{32}$,

and $2^{(p-1)/16} \equiv 1 \pmod{p}$,

let $\pi(\xi) = \Sigma b_v \xi^v \ (v = 0, 1, \ldots, 7)$

be one of the prime factors of p in the field of 16-th roots of 1, in its canonical form, so that

$$\pi \equiv 1 \pmod{2};$$

and let $\pi(\xi)\,\pi(-\xi) = a_0+a_1\xi^2+a_2\xi^4+a_3\xi^6$.

Then $2^{(p-1)/32} \equiv (-1)^y \pmod{p}$:

where (1), when $b \equiv 0 \pmod{32}$, and $d \equiv 0 \pmod{8}$,

$$y \equiv \tfrac{1}{16}a_2+\tfrac{1}{4}(b_2+b_6) \pmod{2};$$

and (2), when $b \equiv 16 \pmod{32}$, and $d \equiv 4 \pmod{8}$,

$$y \equiv \tfrac{1}{16}a_2+\tfrac{1}{4}(b_2-b_6)+1 \pmod{2}.$$

Another application of the author's results is to the formation of the equation of degree l^n satisfied by the m-termed Gaussian periods of p, for any given value of l^n; the coefficients of this equation are rational integral functions of the coefficients of the reciprocal factors of p in the field of l^n-th roots of 1.

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* For the value of $\{2/\mathfrak{p}\}_8$, see *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 251.

† This was discovered by Lt.-Col. A. Cunningham, *Proc. London Math. Soc.*, Ser. 1, Vol. 27, p. 85.

The Aberrations of a Symmetrical Optical System

Mr. T. W. CHAUNDY.

If m, n, $u \equiv ly-mx$, $v \equiv lz-nx$ are the coordinates of a ray before incidence on a coaxial optical system, and if

$$m' = Pm+Qu+\tfrac{1}{2}\{(Am+Bu)(m^2+n^2)+2(Cm+Du)(mu+nv) \\ +(Em+Fu)(u^2+v^2)\}$$

$$u' = Rm+Su+\tfrac{1}{2}\{(Gm+Hu)(m^2+n^2)+2(Im+Ju)(mu+nv) \\ +(Km+Lu)(u^2+v^2)\}, \text{ \&c.,}$$

are its corresponding coordinates to the second approximation on convergence from the optical system, formulæ are obtained expressing the coefficients $A, B \ldots K, L$ in terms of the first order constants of the system and of certain subsidiary systems. Identities connecting these coefficients are established and their derivatives with regard to the powers of the first and last surfaces evaluated.

The Rotation Groups of the Regular Figures in Four or More Dimensions

Mr. E. LINDSAY INCE.

The regular figures in four, and generally in n, dimensions were studied by Stringham (*Amer. Jour. Math.*, Vol. 3, 1880), but so far as the present writer is aware, no investigator has ever considered the rotation groups associated with these figures. In the present paper, the orders of the groups of the five regular figures in four dimensions, and, generally, of the three regular figures in n dimensions ($n > 4$), are obtained. These groups may be regarded as generalisations of the polyhedral groups in three dimensions, but the groups analogous to the dihedral group in three dimensions are excluded from the treatment.

The effect of rotating a regular figure in n dimensions about an axis passing through one of its summits (or other point of symmetry) and the centre of the circumscribing hypersphere, into coincidence with itself, may be regarded from a more fruitful point of view. Such a rotation is equivalent to dissecting the n-dimensional regular figure into its $(n-1)$-dimensional component elements, and then fitting these elements together

in new positions, but in such manner that each element has the same neighbouring elements as before the operation, and so that the summits of the new figure coincide, on the boundary of the hypersphere, with the summits of the original figure.

We may therefore define the order of the group of a regular figure in n dimensions as the number of ways in which the figure can be built up from its component $(n-1)$-dimensional elements, provided always that the figure as a whole occupies a definite fixed position in space, and that its component elements are in the same relative position one to another.

Let us consider, as an example, the 4-dimensional 5-hedroid, composed of 5 tetrahedra, and let us take one of the tetrahedra at random and fix it in a determined position in space which we may call the *base position*. This can be done in 12 ways, since 12 is the number of ways in which the tetrahedron can be made to coincide with itself. But now, once the base tetrahedron is fixed in space, the remaining four tetrahedra (whose positions are predetermined relatively to the base tetrahedron) have their situations completely determined. Now any one of the five tetrahedra may be put into the base position, and hence there are 5×12 or 60 ways of building up the 5-hedroid.

This enumeration is both simple and exhaustive. It is *simple*, for let us suppose that one of the tetrahedra had occupied the same space position twice. Then all the remaining four tetrahedra would have been twice in the same position and orientation, and in particular, the tetrahedron in the base position would have twice taken up one particular configuration, which is untrue. It is *exhaustive*, for let us suppose that one possibility had not been counted, and let us suppose the figure put into that configuration. One or other of the tetrahedra will then occupy the base position. But all the possibilities of arrangement in the base position have been enumerated, which leads to a contradiction. The order of the rotation group of the regular pentahedroid in four dimensions is therefore 60.

The same argument applies to all the other cases, and thus we may conclude that the order of the group of a 4-dimensional regular polyhedroid is the product of the number of 3-dimensional regular polyhedra of which it is built up and the order of the group of a component 3-dimensional polyhedron. In the remaining five cases (in four dimensions), we have

4-dimensional	8-hedroid (8 cubes)	order of group	$8 \times 24 = 192$
,,	16-hedroid (16 tetrahedra)	,,	$16 \times 12 = 192$
,,	24-hedroid (24 octahedra)	,,	$24 \times 24 = 576$
,,	120-hedroid (120 dodecahedra)	,,	$120 \times 60 = 7200$
,,	600-hedroid (600 tetrahedra)	,,	$600 \times 12 = 7200$

As must be the case, the orders of the groups of reciprocal polyhedroids are the same—the 24-hedroid is self-reciprocal.

Generally, in n dimensions ($n > 4$), there are three and only three regular figures, viz. the $(n+1)$-hedroid which is self-reciprocal, and the $2n$-hedroid and the 2^n-hedroid which form a reciprocal pair.

The n-dimensional $(n+1)$-hedroid is composed of $n+1$ $(n-1)$-dimensional n-hedroida: hence

order of group of $(n+1)$-hedroid (in n dimensions)

$= (n+1) \times$ order of group of n-hedroid (in $n-1$ dimensions)

$= (n+1)n \times$ order of group of $(n-1)$-hedroid (in $n-2$ dimensions)

... ...

$= (n+1)\, n\, (n-1) \ldots 6 \,.\, 5 \times$ order of group of tetrahedron (in 3 dimensions)

$= (n+1)\, n\, (n-1) \ldots 6 \,.\, 5 \times 4 \,.\, 3$

$= \frac{1}{2}(n+1)!$.

Similarly,

order of group of $2n$-hedroid (in n dimensions)

$= 2n \times$ order of group of $2(n-1)$-hedroid (in $n-1$ dimensions)

... ...

$= 2n \,.\, 2(n-1)\, 2(n-2) \ldots 2 \,.\, 4 \times$ order of group of cube (in 3 dimensions)

$= 2^{n-1}\, n!$;

order of group of 2^n-hedroid [composed of 2^n $(n-1)$-dimensional n-hedroids]

$= 2^n \times \frac{1}{2} n!$

$= 2^{n-1}\, n!$.

The orders of the groups of each of the two latter figures are the same, since the figures form a reciprocal pair.

Let us consider, in slightly greater detail, the group of the n-dimensional $(n+1)$-hedroid. This figure is characterised by the fact that each summit is equidistant from all other summits. If, therefore, we permute all the diagonals (drawn through the summits and the centre of the hypersphere to the antipodal points on the surface of the hypersphere) among themselves in all possible ways, we rotate the figure into all its possible positions, and also bring it into coincidence with all the possible positions of the reciprocal $(n+1)$-hedroid formed by joining up the antipodal points one to another. All possible *even* permutations of the diagonals give us

the former set of coincidences without bringing in the latter set, in other words they give us the group of the $(n+1)$-hedroid. Consequently, the group of the n-dimensional $(n+1)$-hedroid is simply isomorphous with the alternating group on $n+1$ letters, which has the order $\frac{1}{2}(n+1)!$, in agreement with the previous result.

An Analogue in Space of a Case of Poncelet's Porism

Mr. J. H. GRACE.

The problem of finding a triangle whose vertices lie on one given conic and whose sides touch a second is well known to be poristic. In this paper an analogous problem in n dimensions is considered, viz. :—

To find a simple solid with $n+1$ corners whose vertices lie on a rational curve of degree n and whose faces touch a given (hyper-) quadric surface.

It is shewn that the problem is poristic when n is even, and, in general, admits of a unique solution when n is odd.

The three dimensional case is treated geometrically, *i.e.* it is proved that there is, in general, one tetrahedron whose vertices lie on a given twisted cubic and whose faces touch a given quadric.

Applying this result to the rational quartic curve in four dimensions, I deduce that, if such a curve osculates the hyper-plane at infinity, the hypersphere circumscribing the pentahedron formed by any five osculating hyperplanes of the curve passes through a fixed point. It follows at once that the main problem is poristic in four dimensions.

The general question is then treated algebraically.

A Note on Enumerative Geometry

Mr. J. H. GRACE.

The number N of polygons of n sides whose vertices lie on a given conic, and whose sides touch a given curve of class $r+1$, is found as follows :—

Form the sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots,$

wherein $\lambda_n = r^n - r$ for n odd,

and $\lambda_n = r^n - r^2$ for n even;

deduce the sequence $\mu_1, \mu_2, \ldots, \mu_n, \ldots,$

in which $\mu_1 = 0$, $\mu_2 = 0$,

$$\mu_n = \lambda_n, \text{ for } n \text{ prime},$$

$$\mu_n = \lambda_n - \Sigma\mu_a$$

otherwise, a being any divisor of n: then, finally,

$$N = \mu_n/n.$$

A simple extension of Fermat's theorem provides the verification that N is always an integer.

If we suppose the class curve to be $r+1$ points, N should be the number of ways in which n letters, of $r+1$ different kinds, can be arranged in a ring, no two like letters being adjacent, and periodic arrangements being rejected. This result in combinatory analysis can be verified without difficulty and serves to confirm the rule given above.

LIBRARY

Presents.

BETWEEN December 31st, 1917, and December 31st, 1918, the following presents were made to the Library by their respective authors and publishers :—

de Horsey, Admiral A. R. F.—Draysonia.

Madrid : Junta par Ampliación de Estudios e Investigaciones Científicas. Memoria Correspondiente a los Años 1916 y 1917.

Banos, O. F.—Contribución al Estudio de los Sistemas Lineales de Homografías en E_n. Contribución el Estudio de las Redes de Homografías que contienen la Identiad en E_n. Generalización de un Teorema de Weierstrass.

Moreto, I.—Trabajos de Investigación y Ampliación de Estudios Organisados para el Curso.

Sanz, José Rodriguez.—Groupos de Sustituciones, Tomo 2, Memoria 4*a*.

Tambarrel, J. de Mendezabal.—Tratado Elemental de Goniometria.

Ventosa, Viconte.—Reflexiones acera de la Resolución de las Ecuaciones Algebricas Numéricos par el Método de Graffe.

Viscardini, Mario.—Note on some Correspondence on the Character of Projections, &c.

Washington : U.S. Coast and Geodetic Survey, Department of Commerce. Centennial Celebration.

American Ephemeris and Nautical Almanack for 1920.

Amsterdam : Société Mathématique, Revue semestrielle des Publications Mathématiques, tomes 25, 26.

London : Royal Society. Conjoint Board of Scientific Societies, Minutes of Sixth and Seventh Meetings.

London : Educational Times, vol. 71, nos. 665–668.

London : Educational Times, Math. Questions and Solutions, vol. 5 ; vol. 6, nos. 1–3.

London : H.M. Nautical Almanac Office, Nautical Almanac for 1921.

Madras : Indian Mathematical Society, vol. 9, no. 6 and Index ; vol. 10, nos. 1–5.

Manchester : Journal of the Municipal College of Technology.

Paris : L'enseignement mathématique, 19me année, nos. 4–6 ; 20me année, nos. 1, 2.

Sendai : Tôhuko Mathematical Journal, vol. 12, no. 4 ; vol. 13 ; vol. 14, nos. 1, 2.

Sendai : Tôhuko University, Science Reports, vol. 6, nos. 4–7 ; vol. 7, nos. 1, 2.

Tokyo : Mathematico-Physical Society (Suguku-Buturigahkwai), vol. 9, nos 11–21.

Exchanges.

American Journal of Mathematics, vol. 40, nos. 1–3.

American Mathematical Society, Bulletin, vol. 24, nos. 4–10 ; vol. 25, nos. 1, 2.

American Mathematical Society, Transactions, vol. 19, nos. 1, 2.

American Philosophical Society, Proceedings, vol. 56, nos. 3–7; vol. 57, nos. 1–5; and List of Fellows for 1917.
Amsterdam: Nieuw Archief, deel 12, stuk 3.
Amsterdam: Wiskundige Opgaven, deel 12, stuk 5.
Calcutta: Indian Association for the Cultivation of Science, Proceedings, vol. 3, nos. 5–7.
Calcutta: Mathematical Society, Bulletin, vol. 8, no. 2.
Cambridge Philosophical Society, Proceedings, vol. 19, pt. 4.
Cambridge Philosophical Society, Transactions, vol. 22, nos. 12–14.
Cape Town: South African Association for the Advancement of Science, Journal, vol. 14, nos. 3–12; vol. 15, no. 1.
Catania: Accademia Giornia, Bollettino, fasc. 43.
Dublin: Royal Irish Academy, Proceedings, vol. 43, nos. 1–3.
Edinburgh: Royal Society, Proceedings, vol. 37, no. 4.
Firenze: Biblioteca Nazionale Centrale di Firenze, Bollettino, nos. 205–208, and Alphabetical Index.
Holland: La Société Hollandaise des Sciences à Harlem, Archives Néerlandaises, Tome 4, 2me Livraison; Tome 5, 1re Livraison.
Institute of Actuaries, vol. 51, pt. 1.
Institute of Civil Engineers: List of Fellows, July 1918.
Institution of Naval Architects: Twelve Papers read at the Spring Meetings of the Fifty-Ninth Session.
København: Kgl. Danske Vidensk. Selskab. Mathematisk-fysiske Meddelelser, vol. 1, nos. 3–8.
La Plata: Estudio de las Ciencias físicas y matemáticas, vol. 1, no. 9; vol. 2, nos. 1, 2.
La Plata: Universidad Nacional de la Plata, Memoria Correspondiente a 1916, no. 6.
Livorno: Periódico di Matematica, vol 14, fasc. 1, serie tecnica, vol. 1, no. 5.
London: Nature, vol. 100, nos. 2514–2522; vol. 101; vol. 102, nos. 2531–2564.
London: Philosophical Magazine, vol. 35, nos. 205–210; vol. 36, nos. 211–216.
London: Physical Society, Proceedings, vol. 30, pts. 1–5; vol. 31, pt. 1.
London: Quarterly Journal of Mathematics, vol. 48, no. 2.
London: Royal Astronomical Society, Memoirs, vol. 62, pt. 3.
London: Royal Astronomical Society, Monthly Notices, vol. 78.
London: Royal Astronomical Society, List of Fellows, June 1918.
London: Royal Society, Proceedings, vol. 94; vol. 95, nos. 666, 667.
Manchester: Literary and Philosophical Society, Memoirs and Proceedings, vol. 61, pts. 2, 3.
Mexico: Sociedad Científica "Antonio Alzate," tomo 34, num. 11, 12, and Index.
Paris: Bulletin des sciences mathématiques, tome 41, no. 12; tome 42, nos. 1–9.
Paris: Bulletin de la Société Mathématique de France, tome 45, fasc. 1, 2; et Comptes Rendus de Sciences pour l'année 1917.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 26, fasc. 9–12; vol. 27, 1re semestre, 2me semestre, fasc. 1–8; Rendiconti dell' Adunanza delle Due Classe de 19 Gennaio 1918.
Stockholm: Acta Mathematica, vol. 41, pts. 3, 4.
Toronto: Royal Canadian Institute, vol. 11, pt. 2.
Toulouse: Annales de la Faculté des Sciences de l'Université, tome 6, année 1914.
Washington: National Academy of Sciences, Proceedings, vol. 3, no. 12; vol. 4, nos. 1–11.
Washington: U.S. Naval Observatory, Report for 1917; Publications, Second Series, vol. pts. 3, 4, and Appendix.

OBITUARY NOTICE

PROFESSOR OLAUS HENRICI.

Olaus Magnus Friedrich Erdmann Henrici was born in the year 1840, in Meldorf, on the West Coast of Holstein, where his father held a post in the Danish Civil Service. His father had studied Science and Engineering at the Gewerbeschule in Berlin, which afterwards became the College at Charlottenburg, and he was one of the three who prepared the plans for a canal from Kiel to the mouth of the Elbe. From the age of 9 to 16 Henrici was taught at the Gymnasium in Meldorf. From the age of 16 to 19 he worked as an apprentice in some engineering works at Flensburg. At the age of 19 he went to the Karlsruhe Polytechnicum, where he had the inestimable privilege of coming under the influence of Clebsch, by whose advice he devoted himself entirely to the study of Mathematics. At the age of 22 he went to study under Hesse at Heidelberg, where he obtained the degree of Ph.D. On leaving Heidelberg he went to Berlin, where he studied under Weierstrass and Kronecker. Thence he went to Kiel, where he became a Privat-dozent at the University, but soon found that, being without private means, his prospects were hopeless. So he came to London in 1865, where he struggled for a livelihood, earning a little money by giving private lessons to school boys, until in 1869 he obtained an introduction from Hesse to Sylvester, through whom he became acquainted with Cayley, Clifford, and Hirst. Hirst gave him work in his classes at University College and introduced him to the authorities of Bedford College, where he became Professor of Mathematics. At Easter, 1870, Hirst fell ill and Henrici took over his work, and at the end of the Session was appointed to succeed Hirst, who had resigned.

From 1870 to 1880 he occupied the Chair of Pure Mathematics, which he exchanged in 1880 for that of Applied Mathematics, and this he held until 1884. During his tenure of this last-named Chair he introduced into the course of instruction for English engineers the study of Graphical Statics.

In 1884 he left University College for the Chair of Mechanics and Mathematics at the Central Technical College, where he organised a Laboratory of Mechanics. This was the first of its kind and has been the

model of many others. Almost every piece of apparatus is fitted with a device to make extreme accuracy of measurement possible. The Laboratory contains a large number of calculating machines, planimeters, moment integrators, and, most important of all, the Harmonic Analyser, referred to below. In 1911 his work at the Central Technical College came to an end, when his students presented him with a testimonial and founded the Henrici Medal in his honour, to be given annually at the College for proficiency in Mathematics. He retired to Chandler's Ford in Hampshire, where he devoted himself to gardening. He had been in failing health for some time before his death on the 10th August last.

In 1874 he was elected a Fellow of the Royal Society. From 1882 to 1883 he was a member of the Council of the Royal Society, and from 1882 to 1884 President of the Mathematical Society. In 1884 the University of St. Andrews conferred on him the Honorary Degree of LL.D.

In 1877 he married the daughter of the Rev. Dr. Kennedy and sister of Sir Alexander Kennedy, who survives him. There was one child of the marriage, Major E. O. Henrici of the Royal Engineers.

His mathematical writings are as follows:—

(1) "Bemerkung zu 'Hesse, Zerlegung der Bedingung für die Gleichheit der Hauptaxen eines auf einer Oberfläche zweiten Ordnung liegenden Kegelschnittes in die Summe von Quadraten'," *Crelle's Journal*, Vol. 64, 1865.

(2) "Transformation von Differentialausdrücken erster Ordnung zweiten Grades mit Hülfe der verallgemeinerten elliptischen Coordinaten," *Crelle's Journal*, Vol. 65, 1866.

(3) "Skeleton Structures, especially in Application to Building of Steel and Iron Bridges," Atchley & Co., London, 1866.

(4) "On certain Formulæ concerning the Theory of Discriminants, with Applications to Discriminants of Discriminants and to the Theory of Polar Curves," *Proc. London Math. Soc.*, Ser. 1, Vol. 2, 1869.

(5) On Series of Curves, especially on the Singularities of their Envelopes, with Application to Polar Curves," *Proc. London Math. Soc.*, Ser. 1, Vol. 2, 1869.

(6) "On Congruent Figures" (London Series of Science Class Books, Longmans, Green, & Co., 1879).

(7) "Presidential Address to Section A of the British Association," 1883.

(8) "Über Instrumente zur Harmonischen Analyse" (Catalogue of Exhibits at the Exhibition of Mathematical Apparatus at Munich, 1892).

(9) Notices accompanying Exhibits at the Munich Exhibition (Nos. 47, 90, and 161 in the Catalogue).

(10) "On a New Harmonic Analyser" (*Phil. Mag.*, July, 1894).

11) "An Address on the Use of Vectorial Methods in Physics to Section A of the British Association," 1903.

(12) "On Vectors and Rotors" (jointly with Mr. G. C. Turner, 1903).

(13) "On the Theory of Measurement by Metal Tapes and Wires in Catenary" (Ordnance Survey, 1912, jointly with Major Henrici).

(14) Contributions to the *Encyclopædia Britannica*: six articles on (i) Calculating Machines, (ii) Euclidean Geometry, (iii) Projective Geometry, (iv) Descriptive Geometry, (v) Perspective, (vi) Projection.

No. (1). The first paper in the series deals with an investigation by Hesse, who, starting from the well known fact that the discriminant of the quadratic, which determines the lengths of the principal axes of a conic, whose equation is given in a plane, can be expressed as the sum of two squares, endeavoured to obtain a similar result for a conic when it is given as the intersection of a plane and a quadric surface. Hesse expressed the discriminant successively as the sum of 10, 7, 6 and 5 squares, and then said that it was probably impossible to express it as the sum of two squares. Henrici, making use of Euler's expressions for the direction cosines of three orthogonal axes in terms of three independent quantities, succeeded in expressing the discriminant as the sum of two squares in three different ways.

No. (2). The second paper is an extension to n dimensions of a method employed by Hesse in his lectures on Geometry of Space in the discussion of the problem of the Principal Axes of Conics and Quadrics by the aid elliptic coordinates.

No. (3). The third work on the list is a little book in which Henric advocated the use of pin-jointed structures, the principal advantage of which is that each bar is exposed to tension and compression only in the direction of its longitudinal axis, which is the most favourable condition possible.

Nos. (4) and (5). These papers are related to one another. The work depends on the following important theorem:—

If $F(u, v, w)$ be a rational integral function of u, v, w; if Ω be

the discriminant of this function with regard to u; and if D be the discriminant of Ω with regard to v; then D is of the form AB^2C^3, and $D = 0$ determines w in such a way that Ω has a double root v; and to these values of v, w correspond for $A = 0$ one double root u of $F = 0$; for $B = 0$ two distinct double roots u of $F = 0$; for $C = 0$ one triple root u of $F = 0$.

The orders of A, B, C are determined, and the results applied to determine the order, the class, and the number of singularities of (1) the envelope of a unicursal series of curves, (2) Steinerian curves, (3) polars of curves, (4) mixed polars of curves, and (5) Steinerians of polar curves.

Though it is not mentioned, the work includes the resolution of the c-discriminant locus of the family of curves $f(x, y, c) = 0$ into the envelope-locus, the node-locus, and the cusp-locus, and accounts for the number of times that each factor occurs.

Nos. (6), (7), and [(14) (ii)]. The article on Euclidean geometry [No. (14) (ii)] is a very clear and in the main a descriptive account of Euclid's *Elements* (excluding Books 7, 8, 9, which are arithmetical, and Book 10 which deals with geometrical irrational quantities). Since it was written a good deal of work on the nature and structure of the Fifth Book (on Proportion) has been published. Henrici's own ideas as to the best methods of treating Elementary Geometry will be found in his little book on "Congruent Figures," No. (6), which in a small compass treats of the matter of Euclid's first four books, excepting the theorems relating to areas. It was written with the object of familiarising students from the very first with those modern methods of which the method of projection and the principle of duality are the most fundamental. These methods are introduced in a very simple way. In his preface Henrici says that "the advantages of the method adopted will however be fully appreciated only in their continuation in the second volume which will treat of areas in connection with what Möbius called 'equal figures' and of similar figures." That second volume was never written. It would appear from the concluding remarks of the address [No. (7)] that Henrici had failed to find any treatment that completely satisfied him. He says, "At present it has not even been settled which series of axioms will be ultimately adopted. Of the various systems which have been proposed since the investigations of Riemann and Helmholtz, I may mention here Clifford's suggestion to replace Euclid's Axiom about parallels by the new one, which maintains that in a plane similar figures exist, or, more completely, that at any part in a plane a figure is possible which is similar to any given figure in that plane. This axiom is somewhat startling as long as we have the usual

theory of similar figures in our mind. But the notion of similar figures is truly axiomatic, and it has lately become my conviction that the axiom may be extremely fruitful, and the working out of a syllabus of plane geometry based on it would be very desirable. Possibly many such attempts have still to be made before a new Euclid finds the materials sufficiently prepared for him to raise the hoped for edifice."

The idea of replacing Euclid's postulate of parallels by the postulate that similar figures exist dates back much further than the work of Clifford. It will be found in a lecture by John Wallis in 1663, *Opera Math.*, Vol. 2, pp. 669–678. It reappears in Carnot's *Géométrie de Position* (1803), and in Laplace's *Works*, Vol. 6, p. 472 (1824). Laplace says that the idea of proportionality is a more natural postulate than Euclid's Postulate on Parallels. On the other hand, Bonola (*Non-Euclidean Geometry*, p. 16) says : " The idea of form, independent of the dimensions of the figure, constitutes a hypothesis, which is certainly not more evident than the postulate of Euclid."

The subject seems to be worthy of further research. The idea of similarity is certainly acquired at a very early age. As soon as a child can show that he can recognise a watch from a picture of it he has the idea of similarity. In an observation I once made the age of the child was nine months.

It seems to me a misfortune to science that Henrici did not carry out his idea of writing a book on Similar Figures as he intended.

Nos. (8), (9), (10), and (14) (i). Perhaps the most strikingly original piece of work done by Henrici was the construction of the Harmonic Analyser.

If a function $y = f(x)$ be expressed in a Fourier series

$$y = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi x/c) + \sum_{n=1}^{\infty} B_n \sin(2n\pi x/c),$$

then the coefficients are determined by the formulæ

$$cA_n = 2\int_0^c y \cos(2n\pi x/c)\,dx, \quad cB_n = 2\int_0^c y \sin(2n\pi x/c)\,dx.$$

Clifford showed how the integrals could be obtained mechanically thus :—" If the curve to be analysed be stretched out in the direction of the x to n times its base without altering the y, and then wrapped round a cylinder with circumference c so that it goes n times round, then the orthogonal projection of this curve on the meridian plane which passes through the zero point of the curve will enclose an area which is propor-

tional to B_n. Similarly A_n is got by the aid of a plane perpendicular to the first."

Setting out from this idea Henrici transformed the formulæ for the coefficients into the following :—

$$nA_n = -\frac{1}{\pi}\int \sin(2n\pi x/c)\,dy, \qquad nB_n = \frac{1}{\pi}\int \cos(2n\pi x/c)\,dy,$$

where the integration with regard to y has to be taken over the whole of a curve derived from the original curve by making it continuous. The mechanical evaluation of these integrals is easier than that of the integrals first mentioned. The details of the apparatus will be found in the *Philosophical Magazine* for July 1894. It is sufficient to state here that in the original design the curve was drawn on a cylinder, but by an improvement due to Mr. Sharp, one of Henrici's assistants, it was shown that it might be drawn on a flat sheet of paper. Further improvements in the instrument were made by Coradi.

Part of the instrument is as follows :—

There are wheels of radii proportional to

$$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5}.$$

A wire passes round these, and the coefficients for $n = 1, 2, 3, 4, 5$ are obtained by making the tracer go once round the curve.

Coaxial with these wheels and above them are wheels of radii proportional to

$$\tfrac{1}{6},\ \tfrac{1}{7},\ \tfrac{1}{8},\ \tfrac{1}{9},\ \tfrac{1}{10},$$

respectively.

To determine the coefficients for $n = 6, 7, 8, 9, 10$, the wire must be taken off the first set of wheels and be made to pass round the upper set. The coefficients are then obtained in the same way as for $n = 1, 2, 3, 4, 5$.

No. (9). Of the three exhibits referred to under (9) one is a description of an earlier form of the Harmonic Analyser, already noticed. Another was a model of a surface of the ninth order constructed to illustrate Sylvester's complete invariantive determination of the roots of a quintic in his great memoir on the real and imaginary roots of algebraic equations (*Phil. Trans. Roy. Soc.*, 1864). The third was a model of two confocal hyperboloids connected together so that they could be deformed but always remained confocal hyperboloids. This model had a remarkable history. In the year 1873 Henrici asked one of his pupils to construct a model of a hyperboloid of one sheet by taking three rods to represent three generators of one system, and fastening to them rods to represent generators of

he other system. He believed that the model so constructed would be rigid. To his surprise he found that it was movable. It was then easy to prove that if the centre and the direction of the axes were fixed the hyperboloid could be deformed into any number of confocal hyperboloids. This model was exhibited to the London Mathematical Society in 1874. Soon after this two such models were constructed and so connected that they moved together but always remained confocal. This is the model exhibited at Munich.

Nos. (11) and (12). In his teaching Henrici made great use of Vectorial Analysis, and with the view of introducing its use into English teaching, he delivered an address before Section A of the British Association in 1903 (No. 11). All that remains of his ideas, in addition to this address, is a little book on "Vectors and Rotors," No. (12), based on his lectures to first year students, which was compiled by his assistant, Mr. G. C. Turner. It treats only of the elementary part of the subject, and was intended to be the first part of a book on Vector Analysis, but that book was never completed.

No. (13). This paper, written jointly with Major Henrici, contains a valuable mathematical investigation, due to Henrici, of the Theory of Measurement by Metal Tapes and Wires in Catenary.

Major Henrici, when calculating the length of a geodetic base line at Lossiemouth from the measurements made there in 1909, was unable to find in any former investigation the correction to be applied for the change in shape of the catenary when on any considerable slope. He submitted the point to his father, and the result was the investigation under consideration. This raised the question of the effect produced on the catenary and on the elastic extension of the tape or wire by changes in the tension. The formulæ given make it possible to measure the straight distance between two points on a slope up to 1 in 3, with a resulting error from the causes above mentioned of less than one in a million, and even this small error can be almost entirely eliminated by taking the mean of two measurements with a standard tension applied first at the upper and secondly at the lower end.

No. (14). Of the contributions to the *Encyclopædia Britannica* I have already referred to the first and second. The third and the sixth on Projective Geometry and Projection have a very high reputation as a clear and lucid exposition of the subject. The fourth and fifth are short articles.

Henrici will be remembered chiefly as a great teacher. During his years of struggle (1865–1869) when he was obliged to spend much time

in giving lessons in elementary mathematics to school boys, he had learned to probe the working of the minds of his pupils. This was of great value to him in his subsequent career as a teacher and a writer, for he acquired the faculty of expressing himself with great clearness in both capacities. He published nothing until he felt completely satisfied as to its form. But for this characteristic more of his methods and ideas would have been preserved. I understand that he has left a large amount of manuscript, and it is much to be hoped that some one will be found to go through it with extreme care.

As one of the large number of his pupils I am able to bear testimony to the singular lucidity of his teaching and to his readiness to explain difficulties at all times. With qualities such as these it is easy to understand the mingled respect and affection with which his pupils regarded him. They feel that a great master of his art has passed to his rest.

M. J. M. Hill.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE INHERENTLY CRYSTALLINE STRUCTURE OF A FUNCTION OF ANY NUMBER OF VARIABLES

By W. H. Young *and* Grace Chisholm Young.

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1. In this memoir the new theorems in the Theory of Sets of Points in the Plane and Higher Space, communicated in the companion paper,* are applied to the investigation of the symmetry which must in general hold between the limits of a function at a point, and the discussion of the distribution of the points at which this symmetry is more or less imperfect. Analogy with physical relations suggests the application of the word "crystalline" to such symmetrical relations. It is the structure of a function in the neighbourhood of each point which is the object of our research; not the value at any one point, or in any one region, but all the values in a region of varying shape and size, having the point as boundary point, or, even more generally, all the values at the points of a set of varying form having the point as limiting point.

That the structure of the most general function of one or more real

* "On the Internal Structure of a Set of Points in Space of any Number of Dimensions" (1906), *Proc. London Math. Soc.*, Ser. 2, Vol. 16, pp. 337–351.

variables is inherently crystalline has been shown in previous papers;* the results obtained, except in the case of a single variable, were however not final, and are completed in the present paper. Points at which the structure is absolutely amorphic may indeed occur, but only very exceptionally. In the straight line these exceptional points were shown to be countable; they are the points at which, h being positive and approaching zero as limit, $f(x+h)$ and $f(x-h)$ have not the same set of limits. At the remaining points there is perfect crystalline symmetry.

When we come to a function of two variables, however, the points of the plane are not exhausted by the points of amorphism and those of perfect symmetry. There are degrees of symmetry in the plane and higher space. Moreover there is not one obvious and unique definition of symmetry, as was the case on the straight line. If we abide by the idea of the distinction of right and left, we find, as one of our new results, that *there is no distinction of right and left in the limits of $f(x, y)$, except on a certain countable aggregate of vertical straight lines.* This involves only the variation of x in the function. If y alone varies a similar result holds, the exceptional lines being now horizontal. If, however, we allow both x and y to vary, we shall naturally demand, in the first place, symmetry of the limits of $f(x+h, y+k)$, h and k being both one-signed and approaching zero as limit, that is symmetry in respect of the sets of limits in the four completely open quadrants formed by the parallels to the axes of coordinates through the point. The theorem we obtain is that *there is perfect symmetry with regard to quadrants except at points, forming a set of the first category and content zero, lying on a countable aggregate of non-oscillatory curves.*

The points of amorphism with regard to quadrants (those in which no pair of the four sets of limits are identical) form a countable set.

There are points characterised by intermediate forms of imperfect symmetry, which lie on a countable set of lines parallel to the axes, or are even countable.

We may, however, apply a closer criterion of symmetry, by replacing the quadrants by octants. The main result is then of the same form as before, the eight sets of limits in the octants being identical, except at a

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* W. H. Young, "Some Results in the Theory of Functions of a Real Variable" (1907), *Report of the British Association*, p. 445; "On the Distinction of Right and Left at Points of Discontinuity" (1908), *Quarterly Journal of Mathematics*, Vol. 39, pp. 67–83; "Sulle due funzioni a piu valori costituite dai limiti d'una funzione d'una variabile a destra e a sinistra di ciascun punto" (1908), *Rend. della R. Acc. dei Lincei*, t. 17, pp. 582–587; "On the Discontinuities of a Function of One or More Variables" (1909), *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 117–124.

set of the first category, and content zero, lying on a countable aggregate of non-oscillatory curves. There are again intermediate forms of imperfect symmetry, and points of amorphism, the latter being again countable.

The results are precisely the same when we divide the octants again into halves, or continue further by repeated bisection forming more and more neighbourhoods lying symmetrically round the points under discussion.

Finally, we abandon altogether the idea of a definite region of approach, and obtain the following theorem, showing the extraordinarily deep-seated character of the symmetry which a function presents in its structure.

All the limits of $f(x, y)$ at a point P may be obtained by means of sequences tangential at P to a given half-ray, unless P belongs to a certain set of the first category and content zero, lying on a certain countable set of non-oscillatory curves, independent of the particular half-ray chosen.

2. Let the rational numbers be arranged in the usual manner* in countable order, $r_1, r_2, \ldots$, and let the closed intervals between any two rational numbers be arranged similarly and denoted by $R_1, R_2, \ldots$.† We shall call these the *rational gaps*.

Then, if Π_+ and Π_- are any two closed sets of points on the straight line, and Π_+ is neither identical with Π_-, nor contained in it, there will be a definite *first* rational gap R_i, which contains a point of Π_+ without containing any point of Π_-.

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* The rational proper fractions are first arranged in order, so that those with smaller denominators precede those with greater, and of two with the same denominator, that comes first which has the smaller numerator. We then form a wedge of numbers, of which the series so formed constitutes the first row, and the second row is the same with negative signs. Then follow the same numbers with 1 and -1 added respectively, then with 2 and -2, and so on, and this wedge we read in columns :—

$$\left.\begin{array}{rrrrrl} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3}, & \frac{1}{4}, & \frac{3}{4}, & \ldots \\ -\frac{1}{2}, & -\frac{1}{3}, & -\frac{2}{3}, & -\frac{1}{4}, & \ldots & \\ 1\frac{1}{2}, & 1\frac{1}{3}, & 1\frac{2}{3}, & \ldots & & \\ -1\frac{1}{2}, & -1\frac{1}{3}, & \ldots & & & \end{array}\right\} = \tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{2}, \tfrac{2}{3}, -\tfrac{1}{3}, 1\tfrac{1}{2}, \ldots .$$

† The indices of the end-points serving like the denominator and numerator in the arrangement of the rational proper fractions between 0 and 1 :—

$$R_1 = (r_1, r_2), \quad R_2 = (r_1, r_3), \quad R_3 = (r_2, r_3), \quad \ldots .$$

Now the limits of a perfectly general function $f(x, y)$ of two variables, when the point (x, y) approaches a definite point P along a definite plane set of points having P as limiting point, form a closed set, as is well known and easily proved; they are therefore represented by a closed set of points on the straight line.

Hence given any two forms of such plane sets, independent in form of the particular point P approached—for instance, if one set consists of all the points to the right and the other of all to the left of P*—and denoting by Π_+ the set of all the limits of $f(x, y)$ obtained by approach to P in any manner along points of the first set, and by Π_- the set of all the limits obtained along points of the second set, it follows that each point P at which Π_+ is neither identical with Π_-, nor contained in it, determines a definite integer i, namely the index of the first rational gap R_i which contains a value belonging to Π_+ without containing any value belonging to Π_-. We may class together all the points P, having the same corresponding integer i, into a set S_i, and the whole set S of all the points at which the set Π_+ is neither identical with Π_- nor contained in it will be the sum of the countably infinite set of sub-sets

$$S = S_1 + S_2 + \ldots + S_i + \ldots .$$

It will appear in what follows that this analysis of the set in question is of great utility, and that, in our proofs, we need only discuss the sub-set S_i, instead of the whole set S.

3. We shall use the method of the preceding article first to prove the following theorem :—

Theorem 1.—*If $f(x, y)$ is a perfectly general function of two variables, the limits of $f(x, y)$ as the point (x, y) approaches any definite point P from the right, or from the left, are the same, except for points P lying on a countably infinite set of vertical lines (the axes of x and y being supposed respectively horizontal and vertical).*

Let Π_+ denote the set of all the limits of $f(x, y)$ at a point P on the right, and Π_- that of the limits on the left. Then our enunciation states that Π_+ and Π_- are identical except for points P lying on a certain countably infinite set of vertical lines.

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* The points on the same ordinate as the point P do not of course appear in either set.

We may divide the points P into two classes, those at which Π_+ contains a limit not contained in Π_-, and those at which this is not the case, but Π_- contains a limit not contained in Π_+. It is evidently sufficient to prove the theorem for the former of these classes, at each point of which Π_+ is neither identical with Π_-, nor contained in it.

By the result of the preceding article, our class of exceptional points forms a set S, the sum of a countably infinite aggregate of sets S_i, and it is only necessary to prove that the set S_i lies on a countably infinite set of vertical lines.

Now the definition of the set S_i is that, if P is a point of S_i, the rational gap R_i contains a value belonging to Π_+ but none belonging to Π_-, and i is the least integer having this property.

Each point P of S_i is then the middle point of the right-hand side of a rectangle, with horizontal and vertical sides, containing no point of S_i in its interior. For, if this were not the case, inside any such rectangle we could find a point Q of S_i (Fig. 1); and therefore a point T, so near Q as to lie also inside the rectangle, at which $f(x, y)$ has a value belonging to the gap R_i; this follows from the fact that there is such a value which is a limit of $f(x, y)$ on the right of Q. Hence, letting the rectangle shrink up to the point P, we obtain a sequence of points T, leading to a limit of $f(x, y)$ at P on the left lying in the gap R_i. But this is impossible, since P is a point of the set S_i. Therefore our statement is true.

FIG. 1.

By a theorem* in the theory of plane sets of points given in the companion paper, this sub-set S_i therefore lies on a countable set of vertical lines.

This proves our theorem.

Similarly we have the theorem that *the limits of $f(x, y)$ as (x, y) approaches P from above or from below are the same, except for points lying on a countably infinite set of horizontal lines.*

4. We now proceed to examine the limits of $f(x, y)$ at any point P in the four (completely open) quadrants formed by the (horizontal and vertical) axes of x and y through P. We denote the top right hand, or

* Theorem 1, Cor. 1, § 6.

(+, +)-quadrant as the first, the (+, −)-quadrant as the second, the (−, −) one as the third, and the (−, +) one as the fourth quadrant, numbering the quadrants clockwise (Fig. 2). Corresponding numbers will be used for the neighbourhoods of the point considered, which will be taken to be completely open rectangles, with the point for the appropriate corner.

Fig. 2.

The relations to be obtained result easily from the following auxiliary theorem:—

Auxiliary Theorem.—*If R_i denote any rational gap, and, comparing the limits of $f(x, y)$ at a point P in two definite quadrants, R_i contains a limit in one, but no limit in the other, then in the latter quadrant there is a void at P, that is a neighbourhood of P empty of points having the same property.*

If the two quadrants considered are adjacent, the neighbourhood in question may be taken to include that one of its sides through P which does not lie between the two quadrants considered.

In fact, if the first statement is not true, there must be in every such neighbourhood a point Q, having the same property as P. But since the internal points of a whole rectangle in the first of the two quadrants of Q considered lie in the second of the two quadrants of P, there would then be points in the second of the two quadrants of P, as near Q as we please, at which the value of $f(x, y)$ lies inside the gap R_i. In this way, taking a sequence of such points Q, we obtain a sequence of such points (x, y), leading up to P as limiting point, and such that $f(x, y)$ has along this sequence only limits belonging to the gap R_i. But this is, by hypothesis, impossible. Thus the first statement of the enunciation is true.

The above reasoning still holds if, the two quadrants being adjacent, Q lies on that bounding line of the second quadrant considered which does not bound both the quadrants. Thus the second statement is also proved.

5. The following theorem shows that as a general rule a limit in one quadrant occurs again as a limit in one of the other quadrants.

Theorem 2.—*The points at which the limits of $f(x, y)$ in one of the (completely open) quadrants are not all included among those in one of the remaining quadrants are countable.*

Let us consider the points at which there is a limit in the first quadrant which does not occur in any of the other quadrants.

By the result of § 2, the set S of such points is the sum of a triply infinite series of sub-sets $S_{i,j,k}$, such that, if P is a point of $S_{i,j,k}$, there is in the rational gap R_i a value belonging to the set Π_1 of the limits of $f(x, y)$ at P in the first quadrant, but no value belonging to Π_2; in the rational gap R_j there is a value of Π_1 but none of Π_3, and in R_k there is a value of Π_1, but none of Π_4; moreover i, j, and k are the least integers for which this is true.

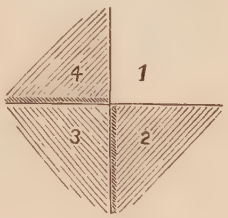


FIG. 3.

By the auxiliary theorem of the preceding article therefore, the set $S_{i,j,k}$ has a void at each of its points P in each of the quadrants, except the first; moreover these voids include the axes between the corresponding quadrants (Fig. 3). The set $S_{i,j,k}$ is therefore* countable. Therefore the whole set of exceptional points in which there is a limit in the first quadrant not occurring in any of the other quadrants is countable. The same being true when we take in turn the remaining quadrants in place of the first, this proves the theorem.

COR. 1.—*The points at which in three of the quadrants the sets of limits are the same, and do not include all the limits in the remaining quadrant, are countable.*

COR. 2.—*The points at which in one, but not in the other, pair of quadrants, the sets of limits of $f(x, y)$ are the same, and do not include all the limits in both the remaining two quadrants, are countable.*

For there will then be a limit, say u, belonging to a set, say Π, of limits in one of the remaining two quadrants, which does not occur in either of the given pair of quadrants. If therefore Π contains a limit which does not occur in the last quadrant (the other of the remaining quadrants mentioned in the enunciation), we have the conditions of the above theorem. If however Π contains no limit which does not occur in the last quadrant, the limit u itself occurs in that last quadrant, and there must be some other limit in that last quadrant which does not occur in Π. Thus again we have the conditions of our theorem.

6. Similarly we have the following theorem :—

THEOREM 3.—*The points at which the limits of $f(x, y)$ in any quad-*

* Theorem 1, Cor. 2, *loc. cit.*

rant do not include all the limits in one of the adjacent quadrants, and at the same time are not all included among those in the other, are countable.

As before it is only necessary to consider the case when the central quadrant of the three is the first; and we see that such points form a set S, the sum of a doubly infinite series of sub-sets $S_{i,j}$, where if P is a point of $S_{i,j}$, R_i is the first rational gap containing a value of Π_2, but not of Π_1, and R_j the first containing a value of Π_1 but not of Π_4.

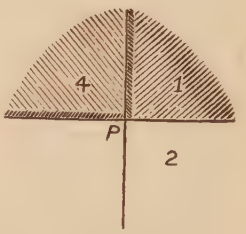


Fig. 4.

By the Auxiliary Theorem of § 4, therefore, the set $S_{i,j}$ has at each of its points P a void in the first quadrant including its vertical boundary, and in the fourth quadrant including its horizontal boundary (Fig. 4), and is therefore a countable set.*

This proves the theorem.

7. Similarly the following theorem is proved:—

Theorem 4.—*The points P at which in two adjacent quadrants the sets of limits of $f(x, y)$ are identical and different from the others, which are not identical, are countable.*

Suppose, for definiteness, that at P the sets Π_2 and Π_3 are identical, and different from Π_1 and Π_4, which are themselves not identical; let us suppose that there is a limit belonging to Π_1 which is not contained in Π_4.

Then, if there is also a limit in Π_1 which is not contained in Π_2, and therefore also not contained in Π_3, we have the case contemplated in Theorem 2.

In the contrary case, Π_1 is contained in Π_2, but is not identical with it. Therefore we have the case contemplated in Theorem 3.

This reasoning is perfectly general, and not dependent on the particular choice of indices here made. Thus our point P necessarily belongs to one of the two sets already proved in Theorems 2 and 3 to be countable.

This proves the theorem.

8. We can now show that there is certainly some form of symmetry

* By the same corollary as in the preceding article.

with respect to the sets of limits in the four quadrants, except at a countable set of points.

THEOREM 5.—*The points P at which all the four sets of limits of $f(x, y)$ in the completely open quadrants are different form a countable set.*

In fact under these circumstances, taking any two adjacent quadrants, there must be one of them in which there is a limit not occurring in the other. Let us suppose, for definiteness, that this is the case with the first and second quadrants, and let us write symbolically

$$\Pi_1 > \Pi_2,$$

or $\Pi_2 < \Pi_1$, to symbolize this relation.

Then since, by Theorem 3, the combination

$$\Pi_1 > \Pi_2 > \Pi_3$$

leads to a countable set of points P, we have only to consider the combination

$$\Pi_1 > \Pi_2 < \Pi_3.$$

Similarly any combination except

$$\Pi_4 < \Pi_1 > \Pi_2 < \Pi_3 > \Pi_4.$$

leads to a countable set. This has to be combined with either

$$\Pi_1 > \Pi_3,$$

or with

$$\Pi_3 > \Pi_1,$$

which, by Theorem 2, each lead to a countable set. The same argument would have held if, of the two first quadrants, it had been the second instead of the first which contained a limit not belonging to the other; we should then have had only to take the first four integers in the order 2, 1, 4, 3.

This proves the theorem.

9. From the preceding theorems we immediately deduce the following:—

THEOREM 6.—*Except at a countable set of points P, we must have for the sets of limits Π_1, Π_2, Π_3, and Π_4 of $f(x, y)$ in the completely open quadrants one of the following combinations:—*

(i) *The four sets Π_1, Π_2, Π_3, and Π_4 are all identical.*

(ii) *Three of the sets are identical, and contain the last.**

(iii) *In one pair of adjacent quadrants the sets of limits are identical, and different from the others, which are identical.*†

(iv) *In a pair of opposite quadrants the sets of limits are identical, and contain both the remaining sets which may be, or not, identical.**

In fact, if three of the sets are identical, and do not contain the last, the point P describes a countable set, by Cor. 1 to Theorem 2.

Thus, if more than two of the sets are identical, we have, barring a countable set, only the combinations (i) and (ii).

If two, and two only, of the sets are identical, these may be adjacent, or opposite. If they are opposite, and do not include all the limits in the remaining quadrants, the point P again describes a countable set by Cor. 2 to Theorem 2. If they are adjacent, and the two remaining sets are not identical, P describes a countable set by Theorem 4.

If no two of the sets are identical, we have the conditions of Theorem 5, leading again to a countable set.

Thus the theorem is proved.

10. The following theorem shows that the combination (iii) of the preceding article is exceptional, all such points P lying on a countable set of parallels to the axes.

Theorem 7.—*The points P at which $f(x, y)$ has the same set of limits in each of a pair of adjacent quadrants, and in the remaining pair the sets are identical, but different from the former set, lie on a countable set of lines parallel to the axis dividing the one pair from the other.*

Thus all such points lie on a countable set of vertical and horizontal straight lines.

Suppose that the dividing axis is vertical, and let us denote by Π_+ and Π_- the sets of limits in the pair of quadrants on the right and left respectively ($\Pi_1 = \Pi_2 = \Pi_+$, $\Pi_3 = \Pi_4 = \Pi_-$).

Then, as in the proof of Theorem 1, we only have to discuss the set S_i of the points P at which i is the least integer such that in the rational gap R_i there is a limit belonging to Π_+ but none belonging to Π_-. By

* This set lies on a countable set of non-oscillatory curves, § 11.

† This set lies on a countable set of straight lines parallel to the axes, § 10.

the Auxiliary Theorem of § 4, therefore, the set S_i has a void in both the quadrants on the left (the third and fourth) including their common boundary. Therefore the set S_i lies on a countable set of vertical lines, by Theorem 1, Cor. 1, of the paper already quoted.

This proves the theorem.

11. That the combinations (ii) and (iv) of § 9 are also exceptional follows at once from the following theorem.

THEOREM 8.—*If in one pair of opposite quadrants the sets of limits of $f(x, y)$ at a point P are identical, and include a limit not contained in one of the two remaining sets, the point P lies on one of a certain countable set of monotone curves.*

Suppose the given opposite quadrants are the second and fourth, and that the corresponding sets of limits Π_2 and Π_4 include a limit not belonging to Π_1. Then, as before, we only have to discuss the set S_i of the points P at which i is the least integer such that, in the rational gap R_i, there is a limit belonging to Π_2 (and therefore to Π_4) but none belonging to Π_1.

By the Auxiliary Theorem of § 4, therefore, the set S_i has a void in the first quadrant, including both the bounding lines, at each of its points P. Thus the set S lies on a countable set of monotone curves, by Theorem 2 of the paper already quoted.

The same result holds if we replace the first by the third quadrant, or if we take instead of the second and fourth, the first and third quadrants for the given pair.

This proves the theorem.

COR.—*The set of points P in question is such that on every horizontal, and on every vertical, line there lies only a countable sub-set, and the monotone curves have no horizontal or vertical parts.*

In fact, since the bounding lines of the void are both included, it is the corollary to Theorem 2 of the paper quoted that here applies.

12. We may therefore complete § 6 by the following theorem :—

THEOREM 9.—*Except at a set of content zero, of the first category, and capable of being arranged on a countable set of monotone curves, including stretches parallel to the axes of coordinates, the four sets of*

limits of $f(x, y)$ at any point P in the quadrants determined by the axes are all the same.

13. Before proceeding further, it will be well to prove the following theorem :—

Theorem 10.—*Among the sets of limits Π_1, Π_2, Π_3, and Π_4 of $f(x, y)$ at a point P in the four quadrants, there is one set which is not contained in any other, except it be identical with it.*

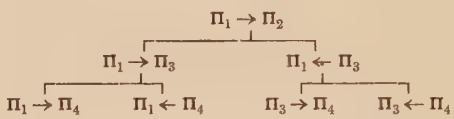
The argument is most easily followed in the form of a sort of genealogical ree. We use the expression

$$\Pi_1 \rightarrow \Pi_2$$

to signify that Π_1 is either identical with Π_2, or that there is a value in Π_1 which is not contained in Π_2. The contrary will be denoted by

$$\Pi_1 \leftarrow \Pi_2,$$

meaning that *every* limit in Π_1 is contained in Π_2 but not *vice versa.* Ascent in the genealogy then means that the circumstances symbolized, as we follow up any branch, hold. Thus we have the following tree, and a similar tree in which Π_1 and Π_2 are interchanged.



The first item of the last line of the tree gives the result that Π_1 has a limit not contained in Π_4, one not contained in Π_3, and one not contained in Π_2, or is identical with one or more of these three sets; thus in this case the set Π_1 satisfies our requirements. The second item of the last line of the tree shows that Π_4, the third that Π_3, and the fourth again that Π_4, satisfies our requirements. The second similar tree adds that Π_2 may satisfy our requirements. The two trees exhaust all the possibilities. Therefore the theorem is true.

14. Let us call such a set of limits Π_i, which is not contained in any other of the sets Π_j, except it be identical with it, *a principal set* Π_i. Then the preceding theorem tells us that *at every point there is at least one principal set* Π_i.

If there is only one principal set, or if, there being more than one,

the principal sets are not all identical, the points in question are countable, by Theorem 2.

If there are *two identical principal sets*, but not more, these must contain the remaining sets, as is shown by the proof of Theorem 2, Cor. 2. If the quadrants corresponding to the two principal sets are adjacent, the points in question will be countable, by Theorem 3, provided the remaining sets are not identical, and will in any case lie on a countable aggregate of straight lines parallel to the axes, by Theorem 7. If, however, the quadrants are opposite, the points lie on a countable aggregate of monotone curves, by Theorem 8.

Finally, if there are *three identical principal sets*, but no more, these must contain the remaining set, since this is not a principal set. The corresponding points again lie on a countable set of monotone curves.

The points that remain, after putting aside such a countable aggregate of monotone curves, are those at which all the four sets Π_1, Π_2, Π_3, and Π_4 are identical, so that they are all principal sets.

In all these results the limits obtained for approach along the axes themselves do not come in, so that none of these results can be deduced from Theorem 1, which tells us that there is no distinction of right and left, except on a countable aggregate of straight lines parallel to the axes.

15. In the form given to our reasoning in §§ 13 and 14, our results are easily extendable to the sets of limits we obtain when we replace the quadrants by octants, or by the equal angles obtained by further bisection of the angles at the point considered.

The definition of a principal set of limits, given in § 11, holds for the sets Π_5, ..., Π_{12}, obtained from the octants, for instance, and the theorem that at every point there is at least one principal set is still true, the proof being precisely analogous to that given in § 13.

It then again follows that if there is only one principal set, the corresponding points are countable, and the same is true if there are two or more, provided they are not all identical.

If there are two identical principal sets, the corresponding points lie on a countably infinite aggregate of non-oscillatory curves, which, in certain cases* are easily seen to be straight lines, and in certain other cases† are perceived to be countable.

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* When the two octants in question are such that each is adjacent to the octant opposite to the other.

† When the two octants are adjacent to one another, or to the same octant.

If there are three identical principal sets, we again get a countable aggregate of non-oscillatory curves.

Thus in general all the eight sets $\Pi_5, \ldots, \Pi_{12}$ *are identical; the exceptional points form a set of the first category and content zero, lying on a countable aggregate of non-oscillatory curves.*

The proofs of these theorems are precisely of the same nature as before, and depend on the theorem of § 9 of the paper already quoted.*

Similar results hold for $\Pi_{13}, \ldots, \Pi_{26}$, corresponding to the division of the right angles into four equal parts, and so on.

16. It might be supposed that our argument would prove that, except on a countable set of non-oscillatory curves, all the limits of $f(x, y)$ at a point P could be obtained by approaching P along some definite straight line. As this result is manifestly untrue, this would be equivalent to a *reductio ad absurdum*, and show that there was some flaw in our previous work.

The fallacy in this supposition will be made plain by the proof of the following theorem.

17. It is perhaps hardly necessary to point out that, if we obtain any unique limit u for $f(x, y)$ by approaching a point P along a certain sequence of points $P_1, P_2, \ldots, P_n, P_{n+1}, \ldots$ this sequence may be assumed, without loss of generality, to be tangential to some one direction. In fact, the lines $PP_1, PP_2, \ldots, PP_n, \ldots$, being infinite in number, determine a certain closed set of limiting lines, from among which we may choose out a definite one, *e.g.*, the first when, starting with the vertical, we rotate the ray in the sense of the hands of a watch. We can then choose out a subsequence of the rays PP_n giving this ray as unique limiting ray; and, approaching P along the corresponding points P_n we get the limit u.

In the following proof, accordingly, we deal with approach to P along a set tangential at P to a certain direction, and we show that we can obtain the same limit by means of a sequence tangential to another direction, except when P lies on a certain countable set of curves. We cannot, however, suppose that the latter sequence lies entirely on the half ray to which it is tangential at P. This explains the fallacy in the supposition alluded to above.

Theorem.—*All the limits of* $f(x, y)$ *at a point* P *may be obtained by*

* "The sub-set of a plane set at the points of which that set has a triangular void lies on a countable aggregate of non-oscillatory curves."

means of sequences tangential at P to a given half ray τ, unless P lies on a certain countable set of non-oscillatory curves, independent of τ.

We shall first take a limit u of $f(x, y)$ obtained by approach to P along a sequence of points $P_1, P_2, \ldots$ tangential to a certain half-ray t, fixed in direction independently of the position of P, and different from the axes, or the lines obtained by continued bisection of the angles formed by these. We shall denote the angles formed by these excepted lines in the usual way by a binary system of indices. The first four quadrants will have the indices 00, 01, 10, and 11; the eight octants in order 000, 001, ..., 111, and so on. At each new stage the indices of the two angles into which any angle is divided are got from the index of the latter by the addition of 0 or 1, according as the new angle comes first or second in the direction of rotation of the hands of a watch. The finite binary indices will in this way be attached to what we have called in the companion paper *the numbered angles of the point P*. The infinite binary indices attach themselves naturally to the various rays other than the rays used in the continued bisection. Thus the infinite index $e_1 e_2 \ldots e_n e_{n+1} \ldots$, where the e's denote zero or unity according to a certain law, will attach itself to the unique ray t, which is generated by all the points common to the angles whose indices are $e_1 e_2$, $e_1 e_2 e_3$, $e_1 e_2 e_3 e_4$, and so on. The index of τ will be denoted by $\epsilon_1 \epsilon_2 \epsilon_3 \ldots$. The set of all the limits of $f(x, y)$ for modes of approach inside any one of the numbered angles will then be denoted by Π with the appropriate index.

Then since the half ray t is internal to the successive numbered angles whose indices are got from $e_1 e_2 \ldots$ by stopping at any particular stage, the value u belongs to the set $\Pi_{e_1 e_2 \ldots e_n}$. Therefore also, unless P lies on a certain countable set of non-oscillatory curves C_n, u also belongs to the set $\Pi_{\epsilon_1 \epsilon_2 \ldots \epsilon_n}$, and this is true for every integer n.

Therefore taking an isosceles triangle, with the angle $\epsilon_1 \epsilon_2 \ldots \epsilon_n$, and equal sides of length 2^{-n}, we can find a point Q_n inside it, such that the value of $f(x, y)$ there differs from u by less than 2^{-n}. The sequence of points $Q_1, Q_2, \ldots$ will then evidently be tangential to the half-ray τ at P, and along this sequence we get the limit u for $f(x, y)$.

This proves the required result when t is not one of the excepted lines. But, by taking a different initial pair of lines instead of the horizontal and vertical, we can make all the excepted lines fall within the indexed angles at every stage. Thus the same result holds as before, when we approach along one of the excepted lines, or along a set tangential to one of them, a new series of non-oscillatory curves C_1', C_2', being determined by the new initial lines.

Thus, unless the point P lies on a certain countably infinite aggregate of curves, namely the set composed of all the sets C_1, C_2, ... and C_1', C_2', ..., all the limits of $f(x, y)$ at P, obtained by approach to P along sequences tangential to any given half-ray t, may be obtained by means of sequences tangential to a fixed half-ray τ arbitrarily chosen beforehand.

This proves the theorem.*

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* This theorem was in embryo already given in "Discontinuities of a Function of One or More Variables", by W. H. Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 124. It was, however, only shown that the exceptional set is a set of the first category. The above result shows that this set is of content zero, and lies on a countable set of non-oscillatory curves.

AN APPLICATION OF CONFORMAL REPRESENTATION TO CERTAIN HYPERGEOMETRIC SERIES

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Introductory.

The method used in this paper appears exactly as here in a paper by Prof. Burnside (*Proc. London Math. Soc.*, Ser. 1, Vol. XXIV, 1893, pp. 187–206), who discussed the conformal representation of certain rectilinear figures, and also remarked that the formulæ for the transformation of the Jacobian elliptic functions can be obtained by means of the representation of rectangles. It consists in taking the area A in a w plane which can be conformally represented on the positive half of a z plane. Crossing the axis of real quantities the function z effects the representation on the negative half of the z plane of a contiguous area A_1 in the w plane. Crossing again to the positive half of the z plane we may or may not have changed the branch of w as a function of z, according as our route does or does not encircle singular points on the real axis, and the function gives the representation of an area A_2 contiguous to A_1, and A_2 may or may not be the same as the original area A, and so on. The device adopted gives the conformal representation of the area obtained by combining two or more of the areas A, A_1, A_2, ... taken necessarily in the order in which they arise in the process above. It has an analytical counterpart given by Klein (*Math. Ann.*, t. XII, pp. 173–176), of which the author has availed himself in a previous paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 15). It has here been applied to the discussion of certain results of Gauss and Kummer with respect to the hypergeometric series.

It is known that certain special cases of the hypergeometric equation are capable of transformation one into another by an algebraic transformation of the independent variable. These are, of course, the cases which

are algebraically integrable, all such cases being reducible to Cases I, II, IV, VI of Schwarz.

In these cases the elements of the hypergeometric equation have definite values. Cases, however, are known in which the desired transformation is possible and some of the elements are quite arbitrary.

Thus, Gauss in his memoir "Determinatio seriei nostrae per aequationem differentialem secundi ordinis" (*Ges. Werke*, t. III), under the title "Quaedam theoremata specialia" (pp. 53–57), gives the three following relations :—

$$(1+y)^{2\alpha} F(2\alpha,\ 2\alpha+1-\gamma,\ \gamma,\ y) = F\left(\alpha,\ \alpha+\tfrac{1}{2},\ \gamma,\ \frac{4y}{(1+y)^2}\right),$$

$$(1+y)^{2\alpha} F(\alpha,\ \alpha+\tfrac{1}{2}-\beta,\ \beta+\tfrac{1}{2},\ y^2) = F\left(\alpha,\ \beta,\ 2\beta,\ \frac{4y}{(1+y)^2}\right),$$

$$F\{\alpha,\ \beta,\ \alpha+\beta+\tfrac{1}{2},\ 4y(1-y)\} = F(2\alpha,\ 2\beta,\ \alpha+\beta+\tfrac{1}{2},\ y).$$

Kummer also has given a large number of such relations in his two papers "Ueber die hypergeometrische Reihe" (*Crelle*, Vol. xv, pp. 64–83 and pp. 127–133). In the earlier paper Kummer's relations refer to cases which have two elements arbitrary, in the latter to cases having only one element arbitrary.

I purpose to show how many such relations can be obtained by the method of conformal representation. We start from the well known result of Schwarz that a curvilinear triangle ABC in a w plane whose angles are $\lambda\pi$, $\mu\pi$, $\nu\pi$ is represented on the positive half of a z plane by means of the relation

$$\tfrac{1}{2}\{w, z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2} + \frac{1-\mu^2}{(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{z(z-1)}\right\}, \qquad \text{(A)}$$

the points $z = 0,\ 1,\ \infty$ corresponding respectively to the vertices ABC, and $\{w, z\}$ being the Schwarzian derivative.

If we cross the real axis between, say, the points $z = 0$, $z = 1$, we obtain the representation on the negative half of the z plane of the triangle ABC', formed by inverting the triangle ABC with respect to the circle AB. We may cross and re-cross the real axis in a similar manner.

In accordance with custom we shade the triangles represented on the positive half of the z plane, and the area consisting of the original triangle

and a contiguous unshaded triangle, say ACC' in Fig. 1, is the fundamental region for the function z.

If λ, μ, ν are the reciprocals of integers z is a uniform function of w, and the symmetrical repetitions of ACC' completely cover the w plane once only. (For an account of these properties, see Forsyth, *Theory of Functions*, Ch. xx.)

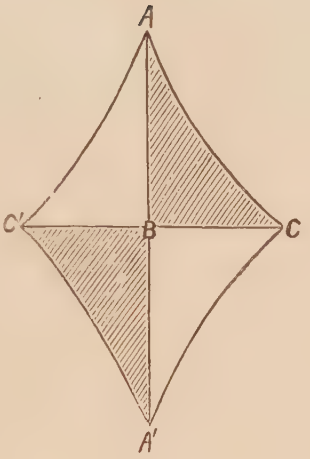

FIG. 1.

In what follows we take $\mu = \frac{1}{2}$.

Let us consider the representation on a z_1 half plane of the triangle ACC'. This is now entirely a shaded area, and the fundamental area for z_1 consists of ACC', and a contiguous unshaded area, say $A'CC'$, The relation between w, z_1 is, of course,

$$\tfrac{1}{2}\{w, z_1\} = \tfrac{1}{4}\left\{\frac{1-4\lambda^2}{z_1^2} + \frac{1-\nu^2}{(z_1-1)^2} + \frac{4\lambda^2+\nu^2-\mu^2-1}{z_1(z_1-1)}\right\}. \qquad \text{(B)}$$

Now take any value of z; to it corresponds one value of w in each of the repetitions of its fundamental area, *i.e.* two values of w in the area $AC'A'C$, and therefore two values of z_1. The relation between z, z_1 is thus quadratic in z_1. Conversely it is of the first degree in z.

This relation which transforms equation (A) into equation (B) can be found by the method of Klein alluded to. This method, however, involves the identification of two expressions, which can usually be done in several ways, and the correct way is only to be determined by trial. By recourse to the geometry of the figure, we may find the required relation with ease.

$z = 0$ corresponds to the points A, A' in the w plane, and therefore to $z_1 = 0$. Further, if z approach the value 0 by any route, the corresponding w points will approach the points A, A', and thus the two corresponding z_1 points will both approach the point $z_1 = 0$. Hence $z_1 = 0$ is a repeated zero of z.

Similarly $z_1 = b$ is a repeated zero of $z-1$, where b is the value of z_1 corresponding to B in the representation of $AC'C$, and the poles of z are simple, viz., $z_1 = 1$, $z_1 = \infty$, so that

$$z = \frac{kz_1^2}{z_1-1}, \qquad z-1 = \frac{k(z_1-b)^2}{z_1-1},$$

whence $$z = \frac{z_1^2}{4(z_1-1)}, \quad z-1 = \frac{(z_1-2)^2}{4(z_1-1)}. \qquad \text{(I)}$$

This is the relation required to transform the equations (A), (B) one into the other.

But the hypergeometric equation whose elements α, β, γ are given by

$$\lambda^2 = (1-\gamma)^2, \quad \mu^2 = (\gamma-\alpha-\beta)^2, \quad \nu^2 = (\alpha-\beta)^2,$$

is connected with the equation

$$\tfrac{1}{2}\{w, z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2} + \frac{1-\mu^2}{(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{z(z-1)}\right\}.$$

In fact, if y_1, y_2 are any two solutions of the hypergeometric equation,

$$w = \frac{Ay_1+By_2}{Cy_1+Dy_2}$$

(see Forsyth, *l.c.*). If then the hypergeometric equations (A), (B) have solutions u_1, u_2; v_1, v_2 respectively, it follows that

$$\frac{u_1}{u_2} = \frac{\alpha v_1+\beta v_2}{\gamma v_1+\delta v_2},$$

or $$u_1 = (\alpha v_1+\beta v_2)\,P, \quad u_2 = (\gamma v_1+\delta v_2)\,P,$$

P being some function of z_1; hence the hypergeometric equation in u, derived from (A), is transformed by the relation (I) into the equation in v derived from (B), save only that the dependent variable is multiplied by a factor P which is a function of z_1.

In fact, the hypergeometric equation derived from (A) is

$$z(z-1)\frac{d^2u}{dz^2} + \{(\tfrac{3}{2}-\lambda)z-(1-\lambda)\}\frac{du}{dz} + \frac{(1-2\lambda+2\nu)(1-2\lambda-2\nu)}{16}u = 0.$$

The substitution $z = \dfrac{z_1^2}{4(z_1-1)}$ transforms this into

$$z_1(z_1-1)^2\frac{d^2u}{dz_1^2} + (z_1-1)\{(\tfrac{3}{2}-\lambda)z_1+2\lambda-1\}\frac{du}{dz_1} + \frac{(1-2\lambda)^2-4\nu^2}{16}z_1u = 0,$$

which may be written

$$z_1(z_1-1)^2\frac{d^2u}{dz_1^2} + (z_1-1)\{(\tfrac{1}{2}+\lambda)z_1+(1-2\lambda)(z_1-1)\}\frac{du}{dz_1} + \frac{(1-2\lambda)^2-4\nu^2}{16}z_1u = 0.$$

Putting $u = (z_1-1)^m v$, the coefficient of the lowest power of (z_1-1) is

$$v\left\{m(m-1)z_1+(\tfrac{1}{2}+\lambda)mz_1+\frac{(1-2\lambda)^2-4\nu^2}{16}z_1\right\},$$

which vanishes when $$m=\frac{1-2\lambda\pm 2\nu}{4}.$$

Taking the lower sign, the equation now becomes

$$z_1(z_1-1)\frac{d^2v}{dz_1^2}+\{(2-2\lambda-\nu)z_1-(1-2\lambda)\}\frac{dv}{dz_1}+\frac{(1-2\lambda)(1-2\lambda-2\nu)}{4}v=0.$$

This is the hypergeometric equation associated with (B). Hence

$$F\left\{\frac{1-2\lambda+2\nu}{4},\ \frac{1-2\lambda-2\nu}{4},\ 1-\lambda,\ \frac{z_1^2}{4(z_1-1)}\right\}$$

$$=(1-z_1)^{\frac{1}{4}(1-2\lambda-2\nu)}\left[AF\left\{\frac{1-2\lambda}{2},\ \frac{1-2\lambda-2\nu}{2},\ 1-2\lambda,\ z_1\right\}\right.$$

$$\left.+Bz_1^{2\lambda}F\left\{\frac{1+2\lambda}{2},\ \frac{1+2\lambda-2\nu}{2},\ 1+2\lambda,\ z_1\right\}\right].$$

Both sides of this equation being capable of expansion in powers of z_1 for sufficiently small values of z_1, and 2λ being supposed not an integer, it follows that $B=0$, $A=1$, and we have, in another form,

$$F\left\{\alpha,\ \beta,\ \alpha+\beta+\tfrac{1}{2},\ \frac{z_1^2}{4(z_1-1)}\right\}=(1-z_1)^{\beta}F\{\alpha+\beta,\ 2\beta,\ 2(\alpha+\beta),\ z_1\}.$$

To obtain our transforming relation we assumed λ, ν to be the reciprocals of integers. In the actual transformation this property is not required; the result is therefore true for arbitrary values of α, β.

I have worked out this case in detail. It will suffice to indicate the transforming relation in other cases.

We might have sought the conformal representation of the triangle $AA'C$. This is fundamentally the same as the preceding case. For purposes of reference, however, I give the transforming relation, which is

$$z=-4z_1(z_1-1),\quad z-1=-(2z_1-1)^2. \qquad (\mathrm{I}a)$$

The hypergeometric equation obtained from this triangle is thus transformable into either of the two preceding cases.

In a similar manner we may obtain relations by means of the triangles

whose angles are $\lambda\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$. In the accompanying figures

$$(a)\ \lambda = \tfrac{1}{4}, \qquad (b)\ \lambda = \tfrac{1}{6}.$$

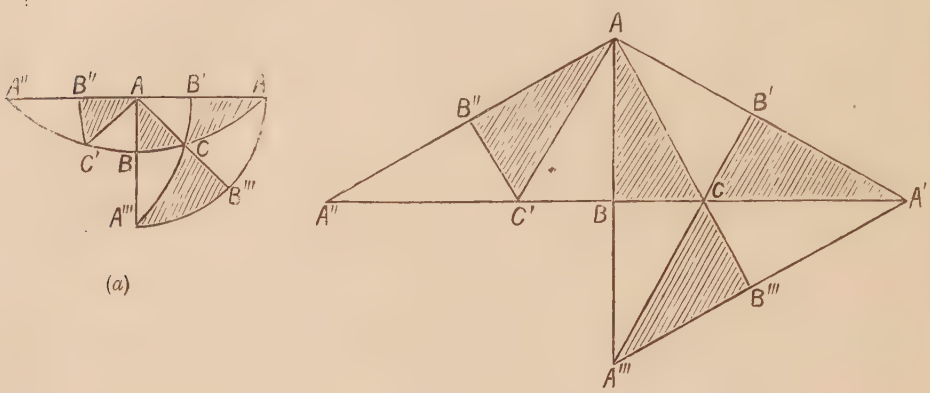

FIG. 2.

First of all we may apply (I) and (Ia), obtaining the conformal representation of triangles whose angles are $2\lambda\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$, and $\lambda\pi$, $\lambda\pi$, $\frac{2}{3}\pi$ respectively.

The transforming relations obtained for various triangles built up of ABC and its repetitions are as below.

Triangles.	Angles.	Transforming Relation.	
ABA'	$2\lambda\pi,\ \frac{\pi}{2},\ \lambda\pi$	$z = -\dfrac{27z_1^2}{(z_1-4)^3}$,	
		$z-1 = -\dfrac{(z_1-1)(z_1+8)^2}{(z_1-4)^3}$	(II)
$AA''A'$	$4\lambda\pi,\ \lambda\pi,\ \lambda\pi$	$z = \dfrac{108z_1^4(z_1-1)}{(z_1^2-16z_1+16)^3}$,	
		$z-1 = -\dfrac{(z_1-2)^2(z_1^2+32z_1-32)^2}{(z_1^2-16z_1+16)^3}$	(III)
$AA'''A'$	$2\lambda\pi,\ 2\lambda\pi,\ 2\lambda\pi$	$z = \dfrac{27z_1^2(z_1-1)^2}{4(z_1^2-z_1+1)^3}$,	
		$z-1 = -\dfrac{27(z_1+1)^2(z_1-2)^2(2z_1-1)^2}{4(z_1^2-z_1+1)^3}$	(IV)
$AA''C$	$3\lambda\pi,\ \lambda\pi,\ \frac{\pi}{3}$	$z = \dfrac{64z_1^3(z_1-1)}{(8z_1-9)^3}$,	
		$z-1 = \dfrac{(8z_1^2-36z_1+27)^2}{(8z_1-9)^3}$.	(V)

We have thus a group of seven distinct triangles, viz., the triangle ABC, and those derived from it in Cases (I), (Ia), (II)–(V). Connected with the representation of each triangle is a hypergeometric series, and these are capable of transformation amongst themselves by means of the relations above. Since only one angle of the original triangle is arbitrary, it follows that only one of the hypergeometric elements is arbitrary.

Finally, we can obtain a similar transformation from the triangle whose angles are $\lambda\pi$, $\frac{1}{2}\pi$, $\frac{1}{4}\pi$. In this case it is possible to apply the combination of case (I) twice in succession. The composite triangle (Fig. 3) has angles $2\lambda\pi$, $\lambda\pi$, $\lambda\pi$, and the transforming relation is

$$z = -\frac{16z_1^2(z_1-1)}{(z_1-2)^4}, \quad z-1 = -\frac{(z_1^2+4z_1-4)^2}{(z_1-2)^4}. \quad \text{(VI)}$$

The derived hypergeometric equation has one arbitrary element.

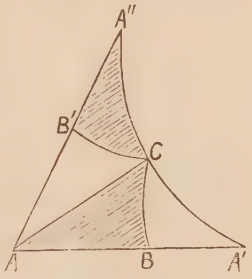

FIG. 3.

I have not been able to discover any other possible combination of triangles, when at the same time we retain at least one of the angles quite arbitrary.

We have not yet, however, exhausted all possibilities. Let us take case (II) above. The equation derived from the triangle whose angles are $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\lambda\pi$, is transformed into the corresponding equation derived from the triangle whose angles are $\frac{1}{2}\pi$, $2\lambda\pi$, $\lambda\pi$ by means of the relation

$$z = -\frac{27z_1^2}{(z_1-4)^3}.$$

But this as an equation in z_1 has three roots, and the transformation makes no distinction amongst them. Let us find the relation between two of them, say ξ_1, ξ_2. We have

$$\frac{(\xi_1-4)^3}{\xi_1^2} = \frac{(\xi_2-4)^3}{\xi_2^2},$$

leading to

$$\xi_1 = \frac{8\xi_2}{\{-1\pm(1-\xi_2)^{\frac{1}{2}}\}^3}. \quad \text{(II}a\text{)}$$

This substitution will therefore transform the hypergeometric equation derived from the triangle whose angles are $\frac{\pi}{2}$, $2\lambda\pi$, $\lambda\pi$ into itself. This result was given by Kummer (*l.c.*, p. 132). Similarly in the case (III) we obtain the substitutions of this character by

combining (I) and (II). In fact, we may write the relation between ξ_1, ξ_2 in this case in the form

$$\frac{\left\{\frac{\xi_1^2}{4(\xi_1-1)}-4\right\}^3}{\left\{\frac{\xi_1^2}{4(\xi_1-1)}\right\}^2}=\frac{\left\{\frac{\xi_1^2}{4(\xi_2-1)}-4\right\}^3}{\left\{\frac{\xi_2^2}{4(\xi_2-1)}\right\}^2}. \qquad (\text{III}a)$$

This may be solved by using (IIa), and gives relations between ξ_1, ξ_2 which transform into itself the hypergeometric equation derived from the triangle whose angles are $4\lambda\pi$, $\lambda\pi$, $\lambda\pi$.

The corresponding substitution from (IV) we deal with in the form

$$\frac{\{4\xi_1(1-\xi_1)-4\}^3}{\{4\xi_1(1-\xi_1)\}^2}=\frac{\{4\xi_2(1-\xi_2)-4\}^3}{\{4\xi_2(1-\xi_2)\}^2}. \qquad (\text{IV}a)$$

This transforms into itself the equation derived from the triangle whose angles are $2\lambda\pi$, $2\lambda\pi$, $2\lambda\pi$.

The corresponding case (Va) admits of no simple expression. For the last case, the relation transforming the equation derived from the triangle whose angles are $2\lambda\pi$, $\lambda\pi$, $\lambda\pi$ into itself can be obtained by taking (VI) in the form

$$\frac{\frac{\xi_1^2}{4(\xi_1-1)}}{\left\{\frac{\xi_1^2}{4(\xi_1-1)}-1\right\}^2}=\frac{\frac{\xi_2^2}{4(\xi_2-1)}}{\left\{\frac{\xi_2^2}{4(\xi_2-1)}-1\right\}^2}. \qquad (\text{VI}a)$$

Cases (I), (Ia), treated in the same manner, each merely provide us with one of the linear transformations of the hypergeometric equation.

I have examined every relation given by Gauss and Kummer (*locis citatis*), and find that the triangles associated with the various hypergeometric series amongst which those relations subsist are all included in the preceding classes. While it might not prove at all easy to obtain by the preceding methods any particular one of the relations given by those authors, remembering that each hypergeometric equation has the twenty-four solutions of Kummer, we see that they do indeed provide a much vaster number of such relations than Kummer gave.

COMBINATIONS DERIVED FROM m IDENTICAL SETS OF n DIFFERENT LETTERS AND THEIR CONNEXION WITH GENERAL MAGIC SQUARES

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In this paper I consider combinations which are derived from m identical sets of n different letters, and shew that there is a one-to-one correspondence between them and squares of order n which are such that the numbers, placed in the cells, have for each row and column a sum m. I generalize this theory and give various theorems and numerical results. The squares in question have been named by me "General Magic Squares," and have been the subject of investigation,* but, in the present paper, squares are considered to be identical if they can be made so by interchange of rows.

The combinations have not, I believe, been treated by any previous investigator.

The problem involved is the enumeration of particular sets of partitions of certain multipartite numbers.

1. If a set of objects be such that p are of one kind, q of a second, r of a third, *et cetera*, the set is said to have a specification

$$(pqr\ldots),$$

which, since p, q, r, ..., may be taken to be numbers in descending order of magnitude with loss of generality, is a partition of the number

$$p+q+r+\ldots.$$

When these objects are distributed into an equal number of boxes of specification

$$(p'q'r'\ldots),$$

one object into each box, the number of the distributions is well known.†

* "Combinatory Analysis," *Camb. Univ. Press*, 1916, Vol. 2, § VIII, Ch. VII.

† *loc. cit.*, Vol. 1, § I, Ch. II.

This number is not altered when the specifications of the objects and of the boxes are interchanged. In fact, the boxes may be also regarded as objects, and the distribution to be a process of forming a set of pairs of objects, one object from each set being a constituent of a pair. Observe that there are no similarities between the objects in the one set and those in the other. If there be any similarities the enumeration of the sets of pairs of objects is a different matter.

Ex. gr.—Denoting the boxes and objects by capital and small letters, respectively, suppose that the boxes and objects both have the specification (111). The distributions are six in number

$$\begin{array}{cccccc} ABC & ABC & ABC & ABC & ABC & ABC \\ abc & acb & bac & bca & cab & cba \end{array}$$

Six sets of pairs of objects can be formed when we regard the boxes as objects.

Suppose now that instead of boxes we have objects of specification (111) precisely similar to the objects a, b, c. We then find the sets of pairs

$$\begin{array}{cccccc} abc & abc & abc & abc & abc & abc \\ abc & acb & bac & bca & cab & cba \end{array}$$

and we note that the sets of pairs shewn by

$$\begin{array}{cc} abc & abc \\ bca & cab \end{array}$$

are identical, each yielding ab, bc, ca.

The number of different sets of pairs is five, not six.

I now discuss the enumeration of the complete pairing of the members of the two identical sets

$$a_1 a_2 a_3 \ldots a_n, \qquad a_1 a_2 a_3 \ldots a_n,$$

each of which has the specification (1^n).

When it is convenient the letters a, b, c, ... will be used.

When $n = 1$, we have one pairing aa.

When $n = 2$, we have the two aa, bb; ab, ab.

A pairing which involves no pair, such as aa or bb, it is convenient to term a "linear pairing."

We thus have, for $n = 2$, two pairings one of which is linear.

When $n = 3$, it has been shewn that there are five pairings, one of which, viz., ab, ac, bc, is linear.

When $n = 4$, denote the number of pairings by

$$\left|\begin{matrix} abcd \\ abcd \end{matrix}\right| \quad \text{or} \quad p_4.$$

If l_4 denotes the number of linear pairings, we reach the relation

$$p_4 = \left|\begin{matrix} a \\ a \end{matrix}\right| \left|\begin{matrix} b \\ b \end{matrix}\right| \left|\begin{matrix} c \\ c \end{matrix}\right| \left|\begin{matrix} d \\ d \end{matrix}\right| + 6 \left|\begin{matrix} a \\ a \end{matrix}\right| \left|\begin{matrix} b \\ b \end{matrix}\right| l_2 + 4 \left|\begin{matrix} a \\ a \end{matrix}\right| l_3 + l_4,$$

or

$$p_4 = 1 + 6l_2 + 4l_3 + l_4,$$

and, generally,

$$p_n = 1 + \binom{n}{2} l_2 + \binom{n}{3} l_3 + \ldots + \binom{n}{n} l_n,$$

which, remembering that

$$l_1 = 0,$$

may be written symbolically

$$p^n = (1 + l)^n,$$

wherein p^s, l^s are symbols for p_s, l_s.

Thence symbolically

$$p = 1 + l,$$

and

$$l^n = (p - 1)^n,$$

leading to the non-symbolic relation

$$l_n = p_n - \binom{n}{1} p_{n-1} + \binom{n}{2} p_{n-2} - \ldots + (-1)^n.$$

We can thus derive the numbers p_n from the numbers l_n and *vice versa*.

Returning to the case $n = 4$, we note that the two objects a may occur in the single pair aa or in the pairs

$$ab,\ ab\,;\quad ac,\ ac\,;\quad ad,\ ad\,;$$

$$ab,\ ac\,;\quad ab,\ ad\,;\quad ac,\ ad\,;$$

and we derive the formula

$$p_4 = p_3 + 3p_2 + 3 \left|\begin{matrix} cd \\ bd \end{matrix}\right|,$$

whence

$$p_4 = 5 + 6 + 6 = 17.$$

The above principle leads to a difference equation for p_n.

For consider the general case

$$\begin{vmatrix} a_1 a_2 a_3 \dots a_n \\ a_1 a_2 a_3 \dots a_n \end{vmatrix} = p_n.$$

We find that

$$p_n = p_{n-1} + \binom{n-1}{1} p_{n-2} + \binom{n-1}{2} \begin{vmatrix} a_2 a_4 a_5 \dots a_n \\ a_3 a_4 a_5 \dots a_n \end{vmatrix}.$$

Moreover, since

$$\begin{vmatrix} a_2 a_4 a_5 \dots a_n \\ a_3 a_4 a_5 \dots a_n \end{vmatrix} = p_{n-3} + (n-3) \begin{vmatrix} a_4 a_5 \dots a \\ a_3 a_5 \dots a^n \end{vmatrix},$$

we reach the formula

$$p_n = p_{n-1} + \binom{n-1}{1} p_{n-2}$$

$$+ \binom{n-1}{2} \{ p_{n-3} + (n-3) p_{n-4} + (n-3)(n-4) p_{n-5} + \dots \},$$

a difference equation which involves $p_0 (= 1)$, $p_1, p_2, \dots, p_n$.

Now, writing

$$q_n = p_{n-3} + (n-3) p_{n-4} + (n-3)(n-4) p_{n-5} + \dots,$$

we find $$q_n = (n-2) q_{n-1} - \binom{n-4}{2} q_{n-3},$$

and thus $$p_n = p_{n-1} + \binom{n-1}{1} p_{n-2} + \binom{n-1}{2} q_n,$$

and finally, a difference equation for p_n which involves six consecutive terms, viz.,

$$(n-3) p_n - (n^2-2n-1) p_{n-1} + (n-1) p_{n-2} + \binom{n-1}{2} (3n-7) p_{n-3}$$

$$-3 \binom{n-1}{3} p_{n-4} - 12 \binom{n-1}{4} p_{n-5} = 0.$$

From these formulæ the following values of p_n, q_n, l_n have been calculated.

n	p	q	l
1	1		0
2	2		1
3	5	1	1
4	17	2	6
5	73	6	22
6	388	23	130
7	2461	109	822
8	18155	618	6202
9	152531	4096	52552
10	1436714	31133	499194
11	14986879	267219	5238370
12	171453343	2557502	60222844

2. These numbers may be studied from another point of view. We may make use of the notation of the Theory of Substitutions. When we denote the pairing by placing the constituents of a pair in the same column, in the manner

$$\begin{matrix} a\,b\,c \\ c\,a\,b \end{matrix}$$

to denote the pairing ac, ab, bc, we observe that

$$\begin{pmatrix} a\,b\,c \\ c\,a\,b \end{pmatrix}$$

denotes a substitution. It is a linear pairing and also a circular substitution of the maximum order 3. In the Theory of Substitutions it may be denoted by

$$(a\,c\,b).$$

Hence we may consider that the substitution $(a\,c\,b)$ also denotes the pairing ac, ab, bc.

The same pairing also corresponds to the substitution $(a\,b\,c)$, because $(a\,c\,b)$, $(a\,b\,c)$ are inverse substitutions.

In general there is a one-to-two correspondence between the related pairing and the two substitutions

$$(a_1 a_2 a_3 \dots a_n), \qquad (a_1 a_n a_{n-1} \dots a_2 a_1),$$

because these are inverse substitutions.

It is thus evident that, regarding the substitutions in the circular notation, we can derive the theory of the pairing.

Ex. gr., of the Order 6 there are substitutions which in circular form correspond to every partition of 6. When the substitution is expressed by a circular substitution of six elements, we find 5! substitutions, giving rise to $\frac{1}{2}.5!$ or 60 distinct pairings. When the substitution involves λ circular substitutions, each of which involves more than two letters, we divide the number of substitutions of the given form by 2^λ to obtain the corresponding number of pairings. The following scheme shews the complete calculation for the Order 6.

Order 6.

Partition.	Type of Substitution.	No. of Substitutions.	Divisor.	No. of Pairings.	
6	(123456)	120	2	60	
24	(12)(3456)	90	2	45	130
33	(123)(456)	40	4	10	
222	(12)(34)(56)	15	1	15	
15	(1)(23456)	144	2	72	
123	(1)(23)(456)	120	2	60	
$1^2 4$	(1)(2)(3456)	90	2	45	
$1^2 2^2$	(1)(2)(34)(56)	45	1	45	
$1^3 3$	(1)(2)(3)(456)	40	2	20	
$1^4 2$	(1)(2)(3)(4)(56)	15	1	15	
1^6	(1)(2)(3)(4)(5)(6)	1	1	1	
		720		388	

verifying the number 388.

The first four partitions of six which do not contain the part unity correspond to the substitutions which displace every letter and yield the linear pairings, shewn above to be 130 in number.

For the order n we have, for the partition

$$1^\alpha 2^\beta 3^\gamma 4^\delta \ldots,$$

a number of substitutions

$$\frac{(\alpha+2\beta+3\gamma+4\delta+\ldots)!}{\alpha!\,\beta!\,\gamma!\,\delta!\ldots\,(2!)^{\beta-\gamma}\,(3!)^{\gamma-\delta}\ldots}.$$

As divisor we have $2^{\gamma+\delta+\dots}$, so that the number of pairings is

$$\frac{(\alpha+2\beta+3\gamma+4\delta+\dots)!}{\alpha!\,\beta!\,\gamma!\,\delta!\dots(2!)^{\beta-\gamma}(3!)^{\gamma-\delta}\dots 2^{\gamma+\delta+\dots}}.$$

The total number of pairings is therefore

$$P_n = \Sigma \frac{(\alpha+2\beta+3\gamma+4\delta+\dots)!}{\alpha!\,\beta!\,\gamma!\,\delta!\dots(2!)^{\beta-\gamma}(3!)^{\gamma-\delta}\dots 2^{\gamma+\delta+\dots}},$$

the summation being for every partition

$$1^\alpha 2^\beta 3^\gamma 4^\delta \dots$$

of n. Also

$$l_n = \Sigma \frac{(2\beta+3\gamma+4\delta+\dots)!}{\beta!\,\gamma!\,\delta!\dots(2!)^{\beta-\gamma}(3!)^{\gamma-\delta}\dots 2^{\gamma+\delta+\dots}},$$

the summation being for every partition

$$2^\beta 3^\gamma 4^\delta \dots$$

of n.

6. There is a third point of view from which the theory of the pairings can be regarded, and it has a larger outlook than those previously examined. We may consider m identical sets of n different letters, and one letter to be taken from each set so as to form an m-ad of letters; this process may be repeated until the sets are exhausted, and we are left with a set of n m-ads of letters. We may enquire as to how many different sets of n m-ads can thus be formed. It is the combination of m-ads which is in question and not permutation between them.

If we look at the elementary case already dealt with for which $n = 3$, $m = 2$, we find the five sets of duads

$$\begin{array}{lll} a^2, & b^2, & c^2 \\ a^2, & bc, & bc \\ ac, & b^2, & ac \\ ab, & ab, & c^2 \\ ab, & ac, & bc \end{array}$$

which are in co-relation with five partitions of the multipartite (here tripartite because $n = 3$) number

viz.,

(200, 020, 002)
(200, 011, 011)
(101, 020, 101)
(110, 110, 002)
(110, 101, 011)

—where observe that 2 occurs in the partitioned multipartite, because here $m = 2$.

The partitions may be defined as being those into three parts for which the weight of each part is 2.

We have, in fact, to find the partitions

$$(\alpha_1\alpha_2\alpha_3,\quad \alpha_4\alpha_5\alpha_6,\quad \alpha_7\alpha_8\alpha_9),$$

such that the following six Diophantine relations are satisfied, viz.,

$$\begin{aligned}
\alpha_1+\alpha_4+\alpha_7 &= 2,\\
\alpha_2+\alpha_5+\alpha_8 &= 2,\\
\alpha_3+\alpha_6+\alpha_9 &= 2,\\
\alpha_1+\alpha_2+\alpha_3 &= 2,\\
\alpha_4+\alpha_5+\alpha_6 &= 2,\\
\alpha_7+\alpha_8+\alpha_9 &= 2,
\end{aligned}$$

and we thus see that there is a one-to-one correspondence between the partitions before us and Magic Squares of a certain nature. If in a square of order n the row and column properties are in evidence but not the diagonal properties, we may call it a General Magic Square of order n.

In the case before us $n = 3$, and the sum of the numbers in any row or column must be $m = 2$. The squares are

2	0	0	2	0	0	1	0	1
0	2	0	0	1	1	0	2	0
0	0	2	0	1	1	1	0	1

1	1	0	1	1	0
1	1	0	0	1	1
0	0	2	1	0	1

If we consider such squares to furnish new squares by interchange of

rows, we could obtain altogether twenty-one squares. We may call the above written squares reduced general magic squares, from which the complete number can be obtained by interchanging rows. We have thus in this case a one-to-one correspondence between the possible pairings of the two sets

$$a\ b\ c$$
$$a\ b\ c,$$

the partitions of the multipartite (222)

into three parts, such that each part is of weight 2, and the reduced general magic squares of order 3, such that the sum of the numbers in any row or column is 2.

There is also a one-to-one correspondence between the sets of pairings, where permutation amongst the pairs is held to give distinct pairings, of the two sets

$$a\ b\ c$$
$$a\ b\ c,$$

the compositions of the partitions of the multipartite

(222)

into three parts, such that each part is of weight 2, and the general (*i.e.*, unreduced) magic squares of order 3, such that the sum of the numbers in any row or column is 2.

Similarly, in the general case, we find a one-to-one correspondence between the sets of m-ads that can be formed from m sets identical with

$$a\ b\ c\ \ldots\ n,$$

the partitions of the n-partite number

$$(m\ m\ m\ \ldots\ m)$$

into n parts, such that each part is of weight m, and the reduced general magic squares of order n, such that the sum of the numbers in any row or column is m.

Also a like correspondence when we take compositions in lieu of partitions.

To make the matter clear it is proper to give other examples of the correspondence.

When $n = 4$, $m = 2$, we have the two sets

$$a\ b\ c\ d$$
$$a\ b\ c\ d$$

giving rise to the 17 pairings

a^2, b^2, c^2, d^2	ab, ab, c^2, d^2	ab, ac, bc, d^2
a^2, b^2, cd, cd	ab, ab, cd, cd	ab, ac, bd, cd
a^2, bd, c^2, bd	ac, ac, b^2, d^2	ab, ad, bd, c^2
a^2, bc, bc, d^2	ac, ac, bd, bd	ab, ad, bc, cd
a^2, bc, cd, bd	ad, ad, b^2, c^2	ac, ad, cd, b^2
	ad, ad, bc, bc	ac, ad, bc, bd

in correspondence with the 17 partitions of (2222)

(2000, 0200, 0020, 0002)
(2000, 0200, 0011, 0011)
(2000, 0101, 0020, 0101)
(2000, 0110, 0110, 0002)
(2000, 0110, 0011, 0101)

(1100, 1100, 0020, 0002)	(1100, 1010, 0110, 0002)
(1100, 1100, 0011, 0011)	(1100, 1010, 0101, 0011)
(1010, 1010, 0200, 0002)	(1100, 1001, 0101, 0020)
(1010, 1010, 0101, 0101)	(1100, 1001, 0110, 0011)
(1001, 1001, 0200, 0020)	(1010, 1001, 0011, 0200)
(1001, 1001, 0110, 0110)	(1010, 1001, 0110, 0101)

and the 17 reduced general magic squares

2000	2000	2000	2000	2000
0200	0200	0101	0110	0110
0020	0011	0020	0110	0011
0002	0011	0101	0002	0101

1100	1100	1010	1010	1001	1001
1100	1100	0200	0101	0200	0110
0020	0011	1010	1010	0020	0110
0002	0011	0002	0101	1001	1001

1100	1100	1100	1100	1010	1010
0110	0101	0101	0110	0200	0101
1010	1010	0020	0011	0011	0110
0002	0011	1001	1001	1001	1001

where it must be borne in mind that squares are to be considered as identical if they can be made the same by interchange of rows. If this condition be not present it will be found that there are 282 different squares. The 17 representative squares have been chosen so that each is vertebrate; that is to say, there is no zero in the principal diagonal. When $n = 3$, $m = 3$, we have the three sets

$$\begin{matrix} a\ b\ c \\ a\ b\ c \\ a\ b\ c \end{matrix}$$

giving rise to the ten sets of 3-ads

$a^3,\ b^3,\ c^3$	$a^2b,\ ab^2,\ c^3$	$ab^2,\ abc,\ ac^2$
$a^3,\ b^2c,\ bc^2$	$a^2b,\ abc,\ bc^2$	$abc,\ abc,\ abc$
	$a^2b,\ ac^2,\ b^2c$	
	$a^2c,\ ac^2,\ b^3$	
	$a^2c,\ abc,\ b^2c$	
	$a^2c,\ ab^2,\ bc^2$	

in correspondence with the ten partitions of (333),

(300, 030, 003)	(210, 120, 003)	(120, 111, 102)
(300, 021, 012)	(210, 111, 012)	(111, 111, 111)
	(210, 102, 021)	
	(201, 102, 030)	
	(201, 111, 021)	
	(201, 120, 012)	

and the ten reduced general magic squares

300	210	201	120
030	120	030	111
003	003	102	102

300	210	201	111
021	111	111	111
012	012	021	111

210	201
021	120
102	012

where again squares are taken to be identical if they can be made so by interchange of rows. Without this condition the total number of squares is seen to be 55. The ten representative squares are vertebrate. The enumeration of general magic squares of order n with an associated number m has been discussed by the author in *Phil. Trans. Roy. Soc.*, A, Vol. 205, 1905, pp. 37–59.

Looking to the 17 magic squares above delineated it will be noticed that those which involve units only correspond to the linear pairings of the two sets. They are six in number in agreement with previous results. In the squares of order 3 last written there is but one square which involves units only in correspondence with the single linear set of triads.

Let the number which enumerates the sets of m-ads that can be formed from m sets of quantities identical with

$$a\ b\ c\ \ldots\ n,$$

as well as the partitions and magic squares associated with them, be denoted by

$$f(m, n).$$

In Arts. 1 and 2 we have discussed the numbers $f(2, n)$.

If we consider the correspondence with partitions, and suppose that the partitions denote symmetric functions, we can construct a function which determines the number $f(m, n)$.

From this point of view the number

$$f(2, n)$$

is equal to the coefficient of the symmetric function

$$\Sigma \alpha_1^2 \alpha_2^2 \dots \alpha_n^2 \equiv (2^n),$$

in the development of the algebraic fraction

$$\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)\dots(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)\dots}.$$

Moreover, if we denote by $g(2, n)$,

the number of linear sets of pairings,

$$g(2, n)$$

is equal to the coefficient of

$$\Sigma \alpha_1^2 \alpha_2^2 \dots \alpha_n^2 \equiv (2^n)$$

in the development of

$$\frac{1}{(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)\dots}.$$

To bring these notions into agreement with previous results we observe that

$$\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)\dots} = 1+(2)+(2^2)+(2^3)+\dots+\text{irrelevant terms},$$

$$\frac{1}{(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)\dots} = 1+(2^2)+(2^3)+6(2^4)+22(2^5)+130(2^6)$$
$$+822(2^7)+\dots+\text{irrelevant terms}.$$

This last result has been taken from the results of Arts. 1 and 2. The direct expansion will be considered presently.

Hence

$$\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)\dots(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)\dots}$$
$$= \{1+(2)+(2^2)+(2^3)+\dots\}\{1+(2^2)+(2^3)+6(2^4)+22(2^5)+\dots\}+\dots,$$

and observing that

$$(2^s)(2^t) = \dots + \binom{s+t}{s}(2^{s+t})+\dots,$$

we readily obtain the expansion

$$1+(2)+2(2^2)+5(2^3)+17(2^4)+73(2^5)+388(2^6)+\ldots,$$

in agreement with Arts. 1 and 2.

When $m=3$, we have to consider the development of the function

$$\frac{1}{(1-a_1^3)(1-a_2^3)(1-a_3^3)\ldots(1-a_1^2a_2)(1-a_1a_2^2)(1-a_1a_3^2)\ldots(1-a_1a_2a_3)(1-a_1a_2a_4)\ldots},$$

and therein the coefficient of the symmetric function

$$\Sigma a_1^3a_2^3a_3^3\ldots a_n^3\equiv(3^n).$$

The number of linear sets of triads $g(3, n)$ is given by the coefficient of the same symmetric function in the expansion of

$$\frac{1}{(1-a_1a_2a_3)(1-a_1a_2a_4)(1-a_2a_3a_4)\ldots}$$

The number of sets of triads in which each triad is of the form

$$a_s^2a_t$$

is given similarly by

$$\frac{1}{(1-a_1^2a_2)(1-a_1a_2^2)(1-a_1a_3^2)\ldots},$$

and, finally, when each triad is of the form

$$a_s^3,$$

the function to be considered is

$$\frac{1}{(1-a_1^3)(1-a_2^3)(1-a_3^3)\ldots}.$$

Since the last function is *effectively*

$$1+(3)+(3^2)+(3^3)+\ldots,$$

it is clear that we may consider separately the function

$$1/(1-a_1^2a_2)(1-a_1a_2^2)\ldots(1-a_1a_2a_3)(1-a_1a_2a_4)\ldots,$$

and then multiply together the two expansions so as to find the desired coefficients.

In general, to find the number $f(m, n)$, we must seek the coefficient of the symmetric function

$$(m^n)$$

in the expansion of a fraction whose numerator is unity and whose denominator is

$$\begin{aligned}&(1-a_1^m)(1-a_2^m)\dots\\ &\times(1-a_1^{m-1}a_2)\dots\\ &\times(1-a_1^{m-2}a_2^2)\dots\\ &\times(1-a_1^{m-2}a_2a_3)\dots\\ &\times\dots,\end{aligned}$$

there being a factor for every term in the product sum h_m homogeneous in respect of the letters

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n.$$

To find the numbers of sets of m-ads which are linear, viz., $g(m, n)$, we take factors corresponding to every term of the elementary symmetric function

$$\Sigma a_1a_2\dots a_m$$

of the same n letters.

The numbers $g(m, n)$ enjoy the property

$$g(m, n) = g(n-m, n).$$

For consider the linear pairing enumerated by $g(2, 5)$, viz.,

$$ab,\ ab,\ cd,\ ce,\ de.$$

We can derive, by dividing each combination of two letters into $abcde$, the linear set of 3-ads

$$cde,\ cde,\ abc,\ abd,\ abc,$$

which is one of the linear sets of 3-ads, enumerated by $g(3, 5)$, derived from the three sets

$$\begin{aligned}&abcde\\ &abcde\\ &abcde\end{aligned}$$

So that $$g(2, 5) = g(3, 5),$$

and generally $$g(m, n) = g(n-m, n).$$

The writer has calculated some of the numbers as shewn in the two tables annexed.

$f(m, n)$

m \ $n=$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	5	17	73
3	1	2	10	93	1317
4	1	3	23	465	
5	1	3	40		
6	1	4	73		
7	1	4	114		

$g(m, n)$

m \ $n=$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2		1	1	6	22	130	822
3			1	1	22	550	16700
4				1	1	130	16700
5					1	1	822
6							1

These numbers may be obtained from the associated generating functions.

The expansion of such a function consists of the homogeneous product sums of the combinations which appear in the denominator factors. These sums are expressible in terms of the sums of the powers of those combinations. An application of Hammond's differential operator

$$D_m^n$$

is then all that is necessary.

This soon becomes laborious. The alternative, failing a difference equation when $m > 2$, is a systematic enumeration of the sets of m-ads. All of the numbers that appear in the tables above have been calculated, for verification, by both methods.

ON THE EFFICIENCY OF A SURFACE OF PRESSURE DISCONTINUITY REGARDED AS A PROPELLER

By W. BURNSIDE.

[Received December 9th, 1916.—Read December 14th, 1916.]

1. The problem of determining the nature of the fluid motion due to a revolving propeller is essentially distinct from that of finding the motion due to a distribution of sources and sinks and solid bodies moving in a fluid. If an ideal closed surface be considered just surrounding the propeller and moving forward with it, then every cubic foot of water which enters this surface will leave it with increased energy, and this increase in energy is due to the action of the propeller on the water. Suppose now that this closed surface is a surface within which external force acts on the fluid in a direction opposite to that in which the surface is moving. The general effect, so far as increasing the energy of the fluid is concerned, will be similar to the effect of the propeller; and, in fact, such a moving region of external force may be regarded as an ideal propeller in which all the mechanism of the screw is dispensed with.

It is certainly beyond the power of mathematical analysis to determine with any completeness the fluid motion close to a rotating propeller, and it would probably be difficult to get a satisfactory solution of the problem when the propeller is replaced by a region of external force.

In the limiting case, however, when the region of external force is replaced by an open surface of pressure discontinuity, it is possible to get definite results. This was done a good many years ago by Mr. R. E. Froude ("On the Part Played in Propulsion by Differences of Fluid Pressure," *Trans. I. N. A.*, 1889), and it was this paper which suggested the present investigation. Mr. Froude's method of establishing the fundamental equation

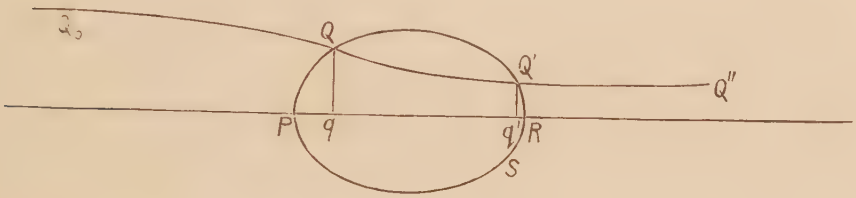
$$\bar{V} = \tfrac{1}{2}(V + V')$$

(see § 4 below) I found difficult to follow, and was thus led to look at the problem from the point of view here developed.

The problems considered from § 7 onwards have no counterpart in Mr. Froude's paper.

The question is treated throughout as one of steady motion in which the region of external force is at rest, and the fluid is flowing through and past it.

2. To fix the ideas, let $PQRS$ be a surface of revolution, axis PR, and suppose that inside this surface a constant external force X per unit mass, parallel to the axis, acts on the fluid. It is proposed to consider what effect this force will have on an otherwise uniform stream, velocity V, parallel to PR, the pressure at infinity being constant and equal to p_0. The motion will be symmetrical about the axis PR.



Let $Q_0QQ'Q''$ represent a stream line which passes through the surface, p be the pressure and v the velocity at any point of the fluid. Then $p+\frac{1}{2}\rho v^2$ is constant along Q_0Q, increases by $\rho X.qq'$ from Q to Q' and is constant along $Q'Q''$. Sufficiently far up stream, the velocity is V and the pressure p_0: and sufficiently far down stream the pressure is p_0. Hence, if V' is the velocity right down stream,

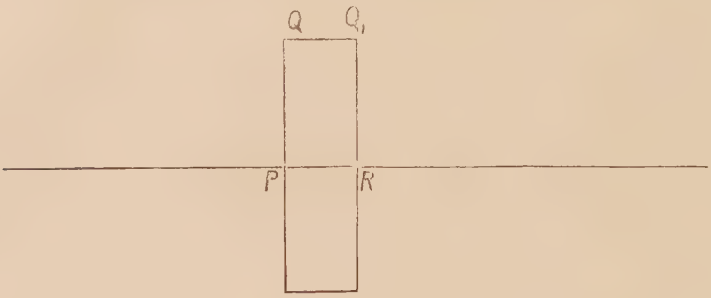
$$V'^2 = V^2+2X.qq'.$$

A tube of flow with $Q_0QQ'Q''$ for its axis will therefore have a smaller cross-section down stream than up stream, and this justifies the general correctness of the above diagram.

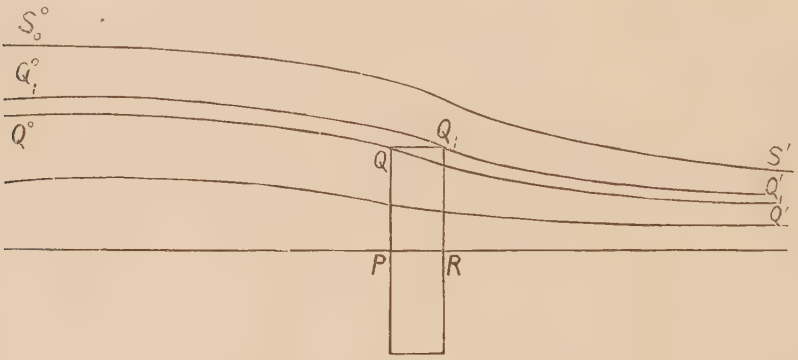
With a surface of the above form, there will be a greater increase of velocity, and therefore a greater contraction, in the tubes of flow which pass through the central parts of the surface than in those which pass further from the centre. In the race, that is in the down stream part of the fluid which passes through the surface, there will at a sufficient distance down stream be a maximum velocity along the axis, and in going outwards from the axis the velocity will fall off till outside the race it is V.

3. To facilitate calculation, suppose that the closed surface is a narrow

slice of a right circular cylinder, the effect produced being kept finite by increasing X as the thickness of the slice diminishes.



If PR is x, the increase in $p+\frac{1}{2}\rho v^2$ for all the stream lines which pass through the surface is the same, viz. ρXx, except for those that abut on the surface between Q and Q_1; and for these the increase falls off from ρXx to 0. Moreover, by diminishing x and increasing X sufficiently, this falling off from Xx to 0 may be made extremely rapid. The general accuracy of the following diagram and description is thus justified.



$Q_1^0 Q_1 Q_1^1$ is the separating stream line dividing those that pass through the surface from those that do not. Outside this the motion is irrotational. Sufficiently far up stream on all the stream lines the velocity is sensibly V. Sufficiently far down stream the velocity from the axis to Q^0QQ' is

$$V' [= \sqrt{(V^2+2Xx)}];$$

between Q^0QQ^1 and $Q_1^0 Q_1 Q_1^1$ it falls from V' to V, and outside $Q_1^0 Q_1 Q_1^1$ it is V.

4. How much of the contraction takes place up stream from the surface and how much down stream it is impossible to say without further

consideration. If S is the section of the surface perpendicular to the axis, and A, B the cross-sections of the tube of flow generated by rotating $Q_1^0 Q_1 Q_1^1$ round the axis, right up stream and right down stream, then, when x is small enough,

$$VA = \bar{V}S = V'B, \tag{i}$$

$\bar{V}$ being the mean velocity parallel to the axis through the surface.

To determine $\bar{V}$ consider a very large sphere with its centre on PR. The motion being steady, the rate at which momentum is leaving this sphere is equal to the resultant force acting on it. If the sphere is large enough, the velocity at every point of it is sensibly V, except where the race cuts it, and there it is V'. Hence, $\bar{V}S$ being the flow along the race, the rate at which momentum leaves the sphere is sensibly $\rho \bar{V} S(V'-V)$. The pressure at the sphere is sensibly p_0, and therefore the resultant pressure on it is sensibly zero. The external force acting on the contents of the sphere is ρSXx, so that

$$\rho \bar{V} S\,(V'-V) = \rho SXx. \tag{ii}$$

5. This equation gives the correct value of $\bar{V}$; but a very cursory examination shews that the statements about the resultant pressure on the sphere and the rate at which momentum leaves it lack justification; and, in fact, they are both untrue.

The irrotational motion outside the separating stream line is finite and continuous everywhere with velocity V at infinity. Hence using polar co-ordinates with an origin on the axis of symmetry, the velocity function is

$$\phi = Vr\cos\theta + \frac{C_0}{r} + \frac{C_1 P_1}{r^2} + \ldots,$$

and the current function

$$\psi = \pi\left(Vr^2\sin^2\theta + 2C_0\cos\theta - \frac{2C_1\sin^2\theta}{r} - \ldots\right).$$

If ψ_0 is the value of ψ for the separating stream line, its equation is

$$\frac{\psi_0}{\pi V} = r^2\sin^2\theta + \frac{2C_0}{V}\cos\theta - \frac{2C_1\sin^2\theta}{Vr} - \ldots.$$

Sufficiently far up stream this is

$$Y^2 = \frac{\psi_0}{\pi V} + \frac{2C_0}{V},$$

and down stream $$Y^2 = \frac{\psi_0}{\pi V} - \frac{2C_0}{V}.$$

But for the separating stream line πY^2 is sensibly A up stream and B down stream. Hence

$$C_0 = \frac{V(A-B)}{4\pi}, \quad \psi_0 = \tfrac{1}{2}V(A+B).$$

Now $$\phi = V\left(r\cos\theta + \frac{A-B}{4\pi r} + \text{terms in } \frac{1}{r^2} \ldots\right),$$

gives $$v^2 = V^2\left(1 - \frac{(A-B)\cos\theta}{2\pi r^2} + \text{terms in } \frac{1}{r^3} \ldots\right)$$

and $$p = p_0 + \frac{\rho V^2(A-B)\cos\theta}{4\pi r^2} + \text{terms in } \frac{1}{r^3} \ldots.$$

Also $$\frac{\partial\phi}{\partial x} = V\left(1 - \frac{(A-B)\cos\theta}{4\pi r^2} + \text{terms in } \frac{1}{r^3} \ldots\right)$$

and $$\frac{\partial\phi}{\partial r} = V\left(\cos\theta - \frac{A-B}{4\pi r^2} + \text{terms in } \frac{1}{r^3}\right).$$

When these expressions are used to calculate the resultant pressure on that part of the sphere $r = R$ which is outside the separating tube of flow, it is found to be

$$-p_0(A-B) - \tfrac{1}{3}\rho V^2(A-B) + \text{terms in } \frac{1}{R}.$$

The resultant pressure on that part of the sphere which is inside the tube of flow obviously differs from $p_0(A-B)$ by a quantity which diminishes indefinitely as R increases. Hence the resultant pressure on the sphere approaches $-\frac{1}{3}\rho V^2(A-B)$ as R increases.

A calculation of the rate at which momentum leaves the part of the sphere outside the separating tube of flow gives the result $-\frac{1}{3}\rho V^2(A-B)$. When x is small enough the same quantity for the part of the spherical surface inside the separating tube is $\rho S\bar{V}(V'-V)$. Equation (ii) is therefore true.

6. When x is sufficiently small the region in which the external force

acts may be regarded as a circular area S over which there is a pressure discontinuity $\varpi\,(=\rho X x)$, constant except very near the edge where it falls rapidly to zero.

The quantities of interest in the propulsion problem are V, V' and S, with the meanings already attached to them; the thrust $P\,(=\varpi S)$, the whole power exerted $H\,(=\frac{1}{2}P\bar{V})$, the useful power PV and the efficiency $e\,(=PV/H)$. From the relations already obtained

$$V'^2-V^2=\frac{2\varpi}{\rho},$$

$$\bar{V}=\tfrac{1}{2}(V+V'),$$

the thrust and the efficiency are determined in terms of S, V, H regarded as data by the equations

$$P^3+2\rho SVHP=2\rho SH^2,$$

and

$$\frac{1-e}{e^3}=\frac{H}{2\rho SV^3}.$$

It may be noted that the efficiency is given in terms of the thrust by the relation

$$\frac{1-e}{e^2}=\frac{P}{2\rho SV^2},$$

and that in any case

$$e=\frac{2V}{V+V'}.$$

7. A real propeller works near the stern of the ship it is driving. A better ideal analogue to it than that considered would be a surface of discontinuous pressure on the down stream side of a fixed obstacle. If the fixed obstacle is a surface of revolution and the centre of S is on its axis, the plane of S being perpendicular to its axis, the motion is still symmetrical; and V, V', $\bar{V}$ being used in the same sense as before the equation (i) still holds.

Moreover, the calculation of the resultant pressure over that part of a large spherical surface which is outside the separating tube of flow, and of the rate at which momentum leaves it, hold as before; but in determining the resultant external force on the contents of the large sphere, the pressure on the obstacle must be taken into account. If this reckoned down stream be Q, equation (ii) is replaced by

$$\rho S\bar{V}(V'-V)=\varpi S-Q.$$

The value of $\bar{V}$ is necessarily less than when there is no obstacle.

In order to get some idea of the effect of the obstacle, it is necessary to determine Q approximately. If V_1 is the mean velocity over S, when there is no pressure discontinuity at S, and $\bar{V}$, as above, when there is, the effect of the pressure discontinuity is to increase the flow through S by $(\bar{V}-V_1)\,S$.

If the ratio of S to the cross-section of the obstacle is comparatively small, and if S is not too near the obstacle, a consideration of the general form of the stream lines will make it clear that *in the neighbourhood of the obstacle* the fluid motion will not be very different from what it would be were the pressure discontinuity replaced by a sink at the middle of S, which withdraws a volume $2(\bar{V}-V_1)\,S$ of fluid per unit time. On this assumption Q can be calculated in the ordinary way.

8. For a workable case it will be assumed that the obstacle is a sphere of radius a, and that the centre of S is at a distance c down stream from the centre of the sphere, the motion being symmetrical about the line joining the centre of the sphere to the centre of S. To justify the assumption made above, a/c must not be too near unity, and $S/\pi a^2$ must be comparatively small.

Now it is easily shewn that the resultant pressure on a sphere of radius a in a stream V, when there is a sink at a distance c down stream from the centre which withdraws a volume $4\pi c^2 U$ of fluid per unit time is

$$\frac{4\pi\rho a^3}{c}\left\{UV+\frac{U^2}{\left(1-\frac{a^2}{c^2}\right)^2}\right\}.*$$

On the assumption that $S/\pi a^2$ is small, the V_1 of the previous statement is $V\left(1-\frac{a^3}{c^3}\right)$, so that

$$U=\frac{\left\{\bar{V}-V\left(1-\frac{a^3}{c^3}\right)\right\}S}{2\pi c^2},$$

and

$$Q=\frac{4\pi\rho a^3}{c}\left\{UV+\frac{U^2}{\left(1-\frac{a^2}{c^2}\right)^2}\right\}.$$

If $S/\pi a^2$ is about $\frac{1}{20}$ and a/c not greater than $\frac{2}{3}$, the second term in Q is

* This result is exact, not approximate.

less than $\frac{1}{60}$th of the first. Hence, as the calculation is only approximate, the second term may be omitted, and

$$Q = \frac{2\rho a^3}{c^3} S \left\{ \bar{V} - V\left(1 - \frac{a^3}{c^3}\right) \right\} V.$$

Using this in (iii),

$$\bar{V}(V' - V) = \frac{\varpi}{\rho} - \frac{2a^3}{c^3} \left\{ \bar{V} - V\left(1 - \frac{a^3}{c^3}\right) \right\} V$$

or $$\bar{V}\left\{ V' - \left(1 - \frac{2a^3}{c^3}\right) V \right\} = \tfrac{1}{2}V'^2 - \left\{ \tfrac{1}{2} - \frac{2a^3}{c^3}\left(1 - \frac{a^3}{c^3}\right) \right\} V^2.$$

The total power H exerted is $\varpi S \bar{V}$, so that

$$\frac{H}{\rho S}\left\{ V' - \left(1 - \frac{2a^3}{c^3}\right) V \right\} = \tfrac{1}{4}(V'^2 - V^2)\left[V'^2 - \left\{ 1 - \frac{4a^3}{c^3}\left(1 - \frac{a^3}{c^3}\right) \right\} V^2 \right].$$

Moreover the efficiency $e\left(= \frac{(\varpi S - Q)V}{\varpi S \bar{V}}\right)$ is as before $\frac{2V}{V + V'}$, and is therefore given by

$$\frac{H}{2\rho S V^3} = \frac{(1-e)\left\{ 1 - e + \frac{a^3}{c^3}\left(1 - \frac{a^3}{c^3}\right) e^2 \right\}}{e^3 \left\{ 1 - \left(1 - \frac{a^3}{c^3}\right) e \right\}}.$$

The efficiency being determined from this equation in terms of the data, H, V, S, the thrust P is given by

$$P = \frac{eH}{V}.$$

To determine the general effect of the obstacle on the efficiency, it is necessary to give a/c a numerical value. The value $\frac{1}{2}$ puts S at a distance behind the sphere equal to the radius, which would seem excessive as compared with the position of a propeller relatively to the ship. This value, however, makes V_1 equal to $\frac{7}{8}V$, and it is doubtful if, in a ship with fine lines, the hull obstructs the flow to the propeller by a greater amount.

When a/c is $\frac{1}{2}$, the equation for the efficiency behind an obstacle which obstructs the free run to the extent of $\frac{1}{8}$th is

$$\frac{H}{2\rho S V^3} = \frac{(1-e)(1 - e + \frac{7}{64}e^2)}{e^3(1 - \frac{7}{8}e)},$$

as compared with $$\frac{H}{2\rho S V^3} = \frac{1-e'}{e'^3},$$

when there is no obstruction.

For given data the roots of the first equation are always less than those of the second, but unless the efficiency is near unity the difference is very small. Thus values of H, V, S which make $e' = \cdot 9, \cdot 8, \cdot 7$ give for e the values $\cdot 89$, $\cdot 79$, and $\cdot 69$ to two figures. The diminution of efficiency due to obstructed flow does not therefore reach $1\frac{1}{2}$ per cent. in the least favourable of these cases.

9. When the fluid is guided to the surface of pressure discontinuity by a rigid tube the problem takes a slightly different form.

Consider a tube of revolution, whose cross-section changes in any way from A to B, the cross-section being constant and equal to A sufficiently far in one direction from the part where the cross-section is changing, and equal to B sufficiently far in the other direction. Suppose that sufficiently far up stream the fluid velocity and pressure are V and p_0 both inside and outside the tube. Then down stream they will be V and p_0 outside and V' and p' inside the tube, where

$$AV = BV',$$

$$p_0 + \tfrac{1}{2}\rho V^2 = p' + \tfrac{1}{2}\rho V'^2.$$

The resultant pressure on that part of a large sphere, whose centre is on the axis, which is outside the tube, is as in § 5

$$-p_0(A-B) - \tfrac{1}{3}\rho V^2(A-B),$$

and the resultant pressure on that part of the sphere inside the tube is

$$p_0 A - p'B.$$

The rate at which momentum leaves the outer part of the sphere is

$$-\tfrac{1}{3}\rho V^2(A-B),$$

and the same quantity for the inner part is

$$\rho V A(V'-V).$$

Hence, if P is the resultant pressure of the tube on the fluid,

$$P = \rho V A(V'-V) - (p_0 - p')B$$

$$= \tfrac{1}{2}\rho(V'-V)^2 B.$$

Up stream, at a sufficient distance from the part of the tube where the

section changes, the tube may be dispensed with; but down stream the pressure inside and outside the tube the pressures are different, and the tube must be regarded as extending indefinitely.

Suppose now that, at a down stream section of the tube where the inside motion is uniform, there is a surface of pressure discontinuity ϖ. Beyond this section the velocity is still V', but the pressure is $p'+\varpi$. If $\varpi = p_0 - p'$, the pressure is the same inside and outside the tube, beyond the surface of discontinuous pressure, and the tube may be dispensed with. Here

$$H = \varpi B V',$$

and the resultant thrust T is given by

$$T = P + \varpi B.$$

The efficiency is

$$\frac{TV}{H} = \frac{V}{V'} \frac{\frac{1}{2}\rho(V'-V)^2 + \frac{1}{2}\rho(V'^2 - V^2)}{\frac{1}{2}\rho(V'^2 - V^2)} = \frac{2V}{V+V'},$$

as was to be expected. When expressed in terms of the same data as in § 6, e is given by

$$\frac{(2-e)(1-e)}{e^3} = \frac{H}{2\rho B V^3}.$$

10. If, instead of supposing that at a down stream section of the tube there is a surface of pressure discontinuity, it is assumed that between two sections suitable external force acts on the fluid, the effect of rotation in the race may be determined.

Taking the axis of z along the axis of a circular tube of radius R, it is easy to verify that the following motion satisfies the dynamical equations, viz.,

$$z < 0, \quad \begin{cases} u = 0, & X = 0, \\ v = 0, \quad p = p', & Y = 0, \\ w = w_0, & Z = 0, \end{cases}$$

$$0 < z < c, \quad \begin{cases} u = -\dfrac{\omega y z}{c}, & \\ v = \dfrac{\omega x z}{c}, \quad p = p' + \rho \dfrac{az}{c} + \dfrac{\rho z^2}{c^2} \displaystyle\int_0^r \omega^2 r\, dr, & \\ w = w_0, & \end{cases}$$

$$X = -\frac{w_0 \omega y}{c},$$

$$Y = \frac{w_0 \omega x}{c},$$

$$Z = a + \frac{2z}{c^2}\int_0^r \omega^2 r\,dr,$$

where ω is a continuous function of r;

$$\begin{array}{lll} & u = -\omega y, & X = 0, \\ z > c, & v = \omega x, \qquad p = p' + \rho a + \rho\int_0^r \omega^2 r\,dr, & Y = 0, \\ & w = w_0, & Z = 0. \end{array}$$

The power exerted viz., $\rho\iiint (Xu + Yv + Zw)\,dx\,dy\,dz$, is

$$\rho w_0 \int_0^R \left[a + \tfrac{1}{2}\omega^2 r^2 + \int_0^r \omega^2 r\,dr\right] 2\pi r\,dr,$$

and the resultant force $\rho\iiint Z\,dx\,dy\,dz$ is

$$\rho\int_0^R \left[a + \int_0^r \omega^2 r\,dr\right] 2\pi r\,dr.$$

If w_0 is the previous V' and πR^2 is B, the pressure at the outer boundary of the race will be the external pressure p_0 if

$$p_0 - p' = \rho\left(a + \int_0^R \omega^2 r\,dr\right),$$

i.e., if

$$a + \int_0^r \omega^2 r\,dr = \tfrac{1}{2}(V'^2 - V^2) - \int_r^R \omega^2 r\,dr,$$

and if this condition is satisfied, the tube beyond the region in which the external force acts may be dispensed with.

In this case

$$H = \rho V'\left\{\tfrac{1}{2}(V'^2 - V^2)B + \int_0^R \left[\tfrac{1}{2}\omega^2 r^2 - \int_r^R \omega^2 r\,dr\right] 2\pi r\,dr\right\},$$

and the total thrust is

$$\rho\left\{V'(V' - V)B - \int_0^R \left[\int_r^R \omega^2 r\,dr\right] 2\pi r\,dr\right\}.$$

The effect of the rotation of the race on the efficiency is given by

$$e = \frac{2V - \dfrac{2V}{BV'(V'-V)}\displaystyle\int_0^R \left[\int_r^R \omega^2 r\,dr\right] 2\pi r\,dr}{V'+V+\dfrac{2}{B(V'-V)}\displaystyle\int_0^R \left[\tfrac{1}{2}\omega^2 r^2 - \int_r^R \omega^2 r\,dr\right] 2\pi r\,dr}.$$

It is easy to verify that so long as ωr vanishes with r, the effect of the rotation, as would be anticipated, is to diminish the efficiency.

11. It is probable that in every case the motion at the boundary of the race is unstable, but it does not seem to me that this vitiates the calculation of the efficiency. If some way down stream the motion breaks up, the distribution of pressure in the neighbourhood of the surface of pressure discontinuity need not thereby be seriously affected.

ORBITS ASYMPTOTIC TO AN ISOSCELES-TRIANGLE SOLUTION OF THE PROBLEM OF THREE BODIES

By DANIEL BUCHANAN.

[Received November 25th, 1916.—Read December 14th, 1916.]

1. *Introduction.*

If two finite spheres of equal masses revolve in circles about their common centre of gravity, then an infinitesimal body, which is subject to their attraction according to the Newtonian law, will describe a periodic orbit under suitable initial conditions. This orbit is in the straight line drawn through the centre of gravity of the finite bodies and perpendicular to the plane of their motion. Since the infinitesimal body remains equidistant from the finite bodies, the periodic solutions defining its motion have been called the "Isosceles-Triangle Solutions of the Problem of Three Bodies." Various cases of these periodic solutions have been discussed,* the simplest case being that in which the equal masses move in circles, and the third body is infinitesimal.

In an article which appeared in the *Proceedings*,† July 1915, the author of the present paper discussed the periodic oscillations near the straight-line orbit mentioned above. The object of the present paper is to determine solutions of the differential equations of motion which will approach this straight-line orbit as the time approaches infinity. Each term of the solutions, however, will be restricted to those having the form

$$e^{kt}P(t) \quad \text{or} \quad e^{-kt}P(t),$$

* Pavanini, "Sopra una Nuova Categoria di Soluzioni Periodiche nel Problema dei Tre Corpi", *Annali di Matematica*, Series 3, Vol. 13, 1907. Macmillan, "An Integrable Case in the Restricted Problem of Three Bodies", *Astronomical Journal*, Nos. 625, 626, 1911. Buchanan, "Isosceles-Triangle Solutions of the Problem of Three Bodies", Moulton's *Periodic Orbits*, Chap. x. Buchanan, "A New Isosceles-Triangle Solution of the Problem of Three Bodies", *Transactions of the American Mathematical Society*, Vol. 16, No. 3, July 1915.

† Buchanan, "Oscillations near one of the Isosceles-Triangle Solutions of the Three-Body Problem". This paper will be cited *Proc.*, followed by a number which refers to an equation in the paper.

where k is a constant, and $P(t)$ is a periodic function of t. Solutions of these types are called, according to Poincaré's definition,* the *asymptotic solutions* of the differential equations considered.

The formal construction of these asymptotic solutions is first made, and their convergence is established later by a reference to a theorem of Poincaré.

The asymptotic orbits constructed in this paper are somewhat analogous to those obtained by Warren,† which are asymptotic to the straight-line equilibrium points of the problem of three bodies.

2. *The Differential Equations of Motion.*

Let the unit of mass be so chosen that the sum of the finite masses is unity, the unit of length so that the distance between them is also unity, and the unit of time so that the gravitational constant is likewise unity. Let the origin of coordinates be taken at the centre of mass and the plane of motion of the finite bodies, m_1, m_2 as the $\xi\eta$-plane. Let the coordinates of m_1, m_2, and the infinitesimal body be denoted by $\xi_1, \eta_1, 0$; $\xi_2, \eta_2, 0$; and ξ, η, ζ respectively.

If m_1 and m_2 are started from the points $\frac{1}{2}, 0, 0$ and $-\frac{1}{2}, 0, 0$ respectively, so that they may move in circles, then

$$\xi_1 = -\xi_2 = \tfrac{1}{2}\cos(t-t_0),$$
$$\eta_1 = -\eta_2 = \tfrac{1}{2}\sin(t-t_0),$$

where t_0 denotes the initial time. The differential equations of motion of the infinitesimal body are [*Proc.* (1)‡]

$$\left.\begin{aligned}
\xi'' &= -\tfrac{1}{2}\xi\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\tfrac{1}{2}\left[\frac{\xi_1}{\rho_1^3}+\frac{\xi_2}{\rho_2^3}\right],\\
\eta'' &= -\tfrac{1}{2}\eta\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\tfrac{1}{2}\left[\frac{\eta_1}{\rho_1^3}+\frac{\eta_2}{\rho_2^3}\right],\\
\zeta'' &= -\tfrac{1}{2}\zeta\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right];\\
\rho_1 &= [(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2]^{\frac{1}{2}},\\
\rho_2 &= [(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2]^{\frac{1}{2}}.
\end{aligned}\right\} \qquad (1)$$

* Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. 1, p. 340.

† L. A. H. Warren, "A Class of Asymptotic Orbits in the Problem of Three Bodies", *American Journal of Mathematics*, Vol. 38, No. 3, pp. 221-249, July 1916.

‡ The constant m in *Proc.* (1) has been put equal to $\frac{1}{2}$.

If the motion of the system is referred to a set of rectangular coordinates rotating about the ζ-axis with uniform unit velocity, that is, if we make in (1) the substitutions

$$\begin{aligned}\xi &= x\cos(t-t_0)-y\sin(t-t_0),\\ \eta &= x\sin(t-t_0)+y\cos(t-t_0),\\ \zeta &= z,\end{aligned}$$

then the differential equations of motion become [*Proc.* (2)]

$$\left.\begin{aligned} x''-2y'-x &= -\tfrac{1}{2}x\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\tfrac{1}{4}\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right],\\ y''+2x'-y &= -\tfrac{1}{2}y\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right],\\ z'' &= -\tfrac{1}{2}z\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right];\\ r_1 &= [(\tfrac{1}{2}-x)^2+y^2+z^2]^{\frac{1}{2}},\\ r_2 &= [(\tfrac{1}{2}+x)^2+y^2+z^2]^{\frac{1}{2}}.\end{aligned}\right\} \qquad (2)$$

If the infinitesimal body is given an initial projection along the z-axis it will remain on this axis, and therefore $x \equiv y \equiv 0$. The differential equation defining its motion then reduces to the simple form

$$z'' = -\frac{8}{[1+4z^2]^{\frac{3}{2}}}.$$

The periodic solution of this equation has already been discussed.*

If the initial values are $z=0$, $\dfrac{dz}{d\tau}=a$ at $\tau=0$, the solution is

$$\begin{aligned} z=\psi \equiv a\sin\tau &+\tfrac{3}{16}a^3[\sin 3\tau-\sin\tau]\\ &+\frac{a^5}{256}[431\sin\tau-192\sin 3\tau+29\sin 5\tau]+\ldots\\ &+a^{2j+1}\sum_{k=1}^{j}A_k^{(2j+1)}\sin(2j+1)\tau+\ldots,\end{aligned}$$

$$\tau=\frac{t-t_0}{[\frac{1}{8}(1+\delta)]^{\frac{1}{2}}}, \qquad \delta=\tfrac{9}{2}a^2-\tfrac{141}{32}a^4+\tfrac{35}{2}a^6+\ldots.$$

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* Buchanan, "The Isosceles-Triangle Solutions of the Problem of Three Bodies" (Part I), Moulton's *Periodic Orbits*, Chap. X.

The constant a is arbitrary, and $a/[\frac{1}{8}(1+\delta)]^{\frac{1}{2}}$ denotes the initial projection of the infinitesimal body from the origin. The series ψ is called *a triply odd series*, since it is odd in a and only odd functions of odd multiples of τ enter the coefficients. Series of this type occur frequently in the sequel, and also series which we shall call *triply even series*, that is, series which are even in a and which have only even functions of even multiples of τ in the coefficients. In both types of series the highest multiple of τ in the coefficient of a^k is k.

3. *The Equations of Variation and their Solution.*

We propose to determine solutions of the differential equations (1) which are asymptotic to the periodic solution

$$x \equiv y \equiv 0, \qquad z = \psi. \tag{3}$$

In the previous paper,* to which reference has been made, periodic solutions of (2) were determined, which are the analytic continuation of this periodic solution (3), when x and y are not identically zero.

The methods of determining the periodic and asymptotic solutions are identical, up to a certain stage. Hence the equations of variation of the present paper and their solutions are the same as those appearing in the previous paper.†

Let us make in (2) the substitutions

$$x = 0+\epsilon\bar{x}, \quad y = 0+\epsilon\bar{y}, \quad z = \psi+w = \psi+\epsilon\bar{w}, \quad t-t_0 = \tau[\tfrac{1}{8}(1+\delta)]^{\frac{1}{2}}, \tag{4}$$

where ϵ is a parameter at present undefined, and $\bar{x}$, $\bar{y}$, and $\bar{w}$ are the new dependent variables. For the meantime, we shall drop the strokes over the x, y, and w, and restore them later. After dividing through by ϵ, and denoting differentiation with respect to τ by a dot over the variable, we obtain from (2) and (4) the differential equations [*Proc.* (5)]

$$\left.\begin{aligned} \ddot{x}-2[\tfrac{1}{8}(1+\delta)]^{\frac{1}{2}}\dot{y}-[\tfrac{1}{8}(1+\delta)+X_0]x &= X \\ &= x[\epsilon X_1+\epsilon^2X_2+\ldots+\epsilon^nX_n+\ldots], \\ \ddot{y}+2[\tfrac{1}{8}(1+\delta)]^{\frac{1}{2}}\dot{x}-[\tfrac{1}{8}(1+\delta)+Y_0]y &= Y \\ &= y[\epsilon Y_1+\epsilon^2Y_2+\ldots+\epsilon^nY_n+\ldots], \\ \ddot{w}+W_0w = W &= \epsilon W_2+\epsilon^2W_3+\ldots+\epsilon^nW_{n+1}+\ldots, \end{aligned}\right\} \tag{5}$$

* Buchanan, *Proceedings*, *loc. cit.*

† There are some slight changes, however, owing (i) to the introduction of the parameter ϵ in the present paper, and (ii) to putting $m_0+\lambda$, of the previous paper, equal to its numerical value $\frac{1}{2}$.

where

$$X_0 = 2-3a^2(1-4\cos 2\tau)+\ldots, \quad Y_0 = -1-\tfrac{3}{2}a^2(1+2\cos 2\tau)+\ldots,$$

$$W_0 = 1-9a^2(\tfrac{1}{2}-\cos 2\tau)+a^4[\tfrac{687}{32}-48\cos 2\tau+\tfrac{177}{8}\cos 4\tau]+\ldots,$$

$$X_1 = -48(1+\delta)\psi w, \quad Y_1 = 12(1+\delta)\psi w,$$

$$X_2 = 8x^2[2-3a^2(7-10\cos 2\tau)+\ldots]-12y^2[2-3a^2(7-5\cos 2\tau)+\ldots]$$
$$-12w^2[2-9a^2(4-5\cos 2\tau)+\ldots],$$

$$Y_2 = -12x^2[2-3a^2(2-5\cos 2\tau)+\ldots]+3y^2[2-a^2(1-10\cos 2\tau)+\ldots]$$
$$+3w^2[2-3a^2(7-10\cos 2\tau)+\ldots],$$

$$W_2 = -x^2[24a\sin\tau+\ldots]+y^2[6a\sin\tau+\ldots]+w^2[18a\sin\tau+\ldots].$$

The terms X_n, Y_n, and W_n are homogeneous polynomials of degree n in x, y, and w, but x and y enter only to even powers. The coefficients in X_n and Y_n are triply even or triply odd series, according as n is even or odd respectively, that is, they are series of the same parity as n, but the coefficients in W_n are series of the opposite parity of n.

For $\epsilon = 1$, the expansions in (5) converge in the spindle obtained by rotating, with respect to the stationary $\xi\eta\xi$-axes, the common part of the two circles with centres at $-\frac{1}{2}$, 0, 0, and $+\frac{1}{2}$, 0, 0, and with radii $1/\sqrt{2}$, about their common chord.* Obviously these series will converge in at least the same region for all ϵ such that $0 < \epsilon \leqslant 1$.

In constructing the asymptotic solutions of (5), it is convenient, although not necessary, to transform the linear terms of the first two equations of (5) to the normal form. In order to obtain the necessary transformation, we require the equations of variation and their solutions. The equations of variation are [*Proc.* (6)]

$$\left.\begin{aligned} &\ddot{x}-2[\tfrac{1}{8}(1+\delta)]^{\frac{1}{2}}\dot{y}-[\tfrac{1}{8}(1+\delta)+X_0]x = 0,\\ &\ddot{y}+2[\tfrac{1}{8}(1+\delta)]^{\frac{1}{2}}\dot{x}-[\tfrac{1}{8}(1+\delta)+Y_0]y = 0,\\ &\ddot{w}+W_0 w = 0. \end{aligned}\right\} \qquad (6)$$

The solutions of the last equation of (6) are not required in this transformation, as this equation is not transformed to the normal form, but they will be given along with the other solutions since they are required later.

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* *Proceedings, loc. cit.*, latter part of Section 3.

The solutions of the first two equations of (6) are [*Proc.* (8)]

$$\left.\begin{aligned} x &= A_1 e^{i\beta_1\tau}u_1 + A_2 e^{-i\beta_1\tau}u_2 + A_3 e^{\beta_2\tau}u_3 + A_4 e^{-\beta_2\tau}u_4, \quad i=\sqrt{-1},\\ y &= ib_1[A_1 e^{i\beta_1\tau}v_1 - A_2 e^{-i\beta_1\tau}v_2] + b_2[A_3 e^{\beta_2\tau}v_3 - A_4 e^{-\beta_2\tau}v_4], \end{aligned}\right\} \tag{7}$$

where the undefined terms of (7) have the following properties. The A_1, A_2, A_3, and A_4 are the constants of integration. The β_1 and β_2 are power series in a^2 with real constant coefficients so determined that u_j and v_j ($j = 1, 2, 3, 4$) shall be periodic with the period 2π in τ. The absolute terms of β_1 and β_2 are $[\sqrt{2}-\frac{3}{8}]^{\frac{1}{2}}$ and $[\sqrt{2}+\frac{3}{8}]^{\frac{1}{2}}$ respectively. The u_j and v_j are power series in a^2, with sums of sines and cosines of even multiples of τ in the coefficients, the highest multiple of τ in the coefficient of a^{2k} being $2k$. In u_1, u_2, v_1, and v_2, the coefficients of the sines are purely imaginary, while the coefficients of the cosines are real. The coefficients of both the sines and cosines in the remaining functions, u_3, u_4, v_3, and v_4 are real. The u_j and v_j satisfy the relations

$$u_1(i) = u_2(-i), \qquad u_3(\tau) = u_4(-\tau),$$

$$v_1(i) = v_2(-i), \qquad v_3(\tau) = v_4(-\tau).$$

Series of these types occur frequently in the sequel, and for brevity we shall denote them symbolically thus :—

$$\begin{aligned} u_1 \text{ or } v_1 \quad &\text{by} \quad (C+iS),\\ u_2 \text{ or } v_2 \quad &\text{by} \quad (C-iS),\\ u_3 \text{ or } v_3 \quad &\text{by} \quad (c+s),\\ u_4 \text{ or } v_4 \quad &\text{by} \quad (c-s). \end{aligned}$$

Subscripts, superscripts, and strokes may be used, but two series which differ only in sign will have the same symbols except for this change of sign. Thus

$$(C_j^{(k)}+iS_j^{(k)}) \quad \text{and} \quad (C_j^{(k)}-iS_j^{(k)})$$

represent power series of the same form as u_1 and u_2, the only difference between them being in the sign of i, which occurs in the coefficients of the sines. Similarly,

$$(c_j^{(k)}+s_j^{(k)}) \quad \text{and} \quad (c_j^{(k)}-s_j^{(k)})$$

are power series of the same form as u_3 and u_4, and they differ from one another only as indicated, that is, in the signs of the sines.

Since the solutions (7) are multiplied by arbitrary constants, the value of each u_j at $\tau = 0$ has been chosen to be unity, and the b_1 and b_2 determined so that each $v_j(0)$ is likewise unity. It is found that b_1 and b_2 are power series in a^2 with constant coefficients, and that the absolute terms of these series are

$$-\frac{[2+\frac{7}{4}\sqrt{2}]}{[\sqrt{2}-\frac{3}{8}]^{\frac{1}{2}}} = -4{\cdot}39\ldots,$$

and

$$\frac{[2-\frac{7}{4}\sqrt{2}]}{[\sqrt{2}+\frac{3}{8}]^{\frac{1}{2}}} = -0{\cdot}350\ldots,$$

respectively. The determinant Δ of the solutions (7) and their first derivatives is a constant,* and its value is [*Proc.* (10)†]

$$\Delta = -i[10\sqrt{119}+\text{terms in } a^2]. \tag{8}$$

This determinant is different from zero for $|\ a\ |$ sufficiently small, and consequently the solutions (7) constitute a fundamental set..

The solutions of the last equation of (6) are [*Proc.* (7)]

$$w = B_1\phi+B_2(\chi+A\tau\phi),$$

$$\phi = \cos\tau+\tfrac{9}{16}a^2[\cos 3\tau-\cos\tau]+\ldots,$$

$$\chi = \sin\tau+\tfrac{9}{16}a^2[\sin 3\tau+5\sin\tau]+\ldots,$$

$$A = -\tfrac{9}{2}a^2+\tfrac{141}{16}a^4+\ldots,$$

where the B_1 and B_2 are the constants of integration. The solutions ϕ and χ have been so determined that the determinant $\phi\dot{\chi}-\dot{\phi}\chi$ has the value unity.

4. *Transformation to the Normal Form.*

Normal variables η_1, η_2, η_3, and η_4 are introduced in the first two equa-

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* Moulton's *Periodic Orbits*, § 18.

† The value of Δ in *Proc.* (10) was computed for the rows $x, \dot{x}, y, \dot{y}$, but in the transformation (9) it is advisable to take the determinant having the rows $x, y, \dot{x}, \dot{y}$, in order that the second subscript on the minors in (10) shall correspond with the subscript of the normal variable. This accounts for the minus sign which occurs in (8) and not in *Proc.* (10).

tions of (6) by the transformation,*

$$\left.\begin{aligned}
x &= \eta_1 u_1 &&+\eta_2 u_2 &&+\eta_3 u_3 &&+\eta_4 u_4,\\
y &= \eta_1 i b_1 v_1 &&-\eta_2 i b_1 v_2 &&+\eta_3 b_2 v_3 &&-\eta_4 b_2 v_4,\\
\dot{x} &= \eta_1[i\beta_1 u_1+\dot{u}_1] &&-\eta_2[i\beta_1 u_2-\dot{u}_2] &&+\eta_3[\beta_2 u_3+\dot{u}_3] &&-\eta_4[\beta_2 u_4-\dot{u}_4],\\
\dot{y} &= \eta_1 b_1[-\beta_1 v_1+i\dot{v}_1] &&-\eta_2 b_1[\beta_1 v_2+i\dot{v}_2] &&+\eta_3 b_2[\beta_2 v_3+\dot{v}_3] &&+\eta_4 b_2[\beta_2 v_4+\dot{v}_4].
\end{aligned}\right\} \tag{9}$$

The determinant of this transformation is Δ of equation (8), and differs from zero when $|\,a\,|$ is sufficiently small. If we solve (9) for η_1, η_2, η_3, and η_4 in terms of x, y, $\dot{x}$, and $\dot{y}$, then differentiate the resulting equations with respect to τ, and eliminate $\ddot{x}$ and $\ddot{y}$ by means of the differential equations (5), we obtain the differential equations in the normal form. They are

$$\left.\begin{aligned}
\dot{\eta}_1 - i\beta_1\eta_1 &= \frac{\Delta_{31}}{\Delta}X+\frac{\Delta_{41}}{\Delta}Y,\\
\dot{\eta}_2 + i\beta_1\eta_2 &= \frac{\Delta_{32}}{\Delta}X+\frac{\Delta_{42}}{\Delta}Y,\\
\dot{\eta}_3 - \beta_2\eta_3 &= \frac{\Delta_{33}}{\Delta}X+\frac{\Delta_{43}}{\Delta}Y,\\
\dot{\eta}_4 + \beta_2\eta_4 &= \frac{\Delta_{34}}{\Delta}X+\frac{\Delta_{44}}{\Delta}Y,
\end{aligned}\right\} \tag{10}$$

where Δ_{jk} is a minor of an element in the determinant of the transformation (9), j referring to the row, and k to the column.

If the coefficients of X and Y in (10) are expressed in the symbolic form adopted for u_k and v_k, these equations have the form

$$\left.\begin{aligned}
\dot{\eta}_1 - i\beta_1\eta_1 &= (S+iC)X+(\bar{C}+i\bar{S})Y,\\
\dot{\eta}_2 + i\beta_1\eta_2 &= (S-iC)X+(\bar{C}-i\bar{S})Y,\\
\dot{\eta}_3 - \beta_2\eta_3 &= (s+c)X+(\bar{c}+\bar{s})Y,\\
\dot{\eta}_4 + \beta_2\eta_4 &= (s-c)X+(\bar{c}-\bar{s})Y,\\
\ddot{w}+W_0 w &= W.
\end{aligned}\right\} \tag{11}$$

The last equation of (11) is repeated from (5).

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* See *Mécanique céleste*, Vol. 1, p. 336.

5. *Formal Construction of Asymptotic Solutions.*

We shall first make the formal construction of the asymptotic solutions which approach zero as τ approaches $+\infty$. Then we shall show that the asymptotic solutions which approach zero as τ approaches $-\infty$ can be obtained directly from these.

$$\text{Let} \qquad \eta_j = \sum_{k=0}^{\infty} \eta_j^{(k)} \epsilon^k \ (j = 1, 2, 3, 4), \quad w = \sum_{k=0}^{\infty} w^{(k)} \epsilon^k, \tag{12}$$

where the $\eta_j^{(k)}$ and $w^{(k)}$ are to be determined from the differential equations (11), subject to the two conditions:

(A) that they shall contain the factor $e^{-k\beta_2\tau}$, where k is a positive integer; and

(B) that $x = b$ at $\tau = 0$.

When the expansions for the η_j are substituted in (9), it follows that x and y are likewise power series in ϵ of the form

$$x = \sum_{k=0}^{\infty} x^{(k)} \epsilon^k, \qquad y = \sum_{k=0}^{\infty} y^{(k)} \epsilon^k, \tag{13}$$

where

$$\left.\begin{aligned} x^{(k)} &= \eta_1^{(k)} u_1 + \eta_2^{(k)} u_2 + \eta_3^{(k)} u_3 + \eta_4^{(k)} u_4, \\ y^{(k)} &= \eta_1^{(k)} i b_1 v_1 - \eta_2^{(k)} i b_1 v_2 + \eta_3^{(k)} b_2 v_3 - \eta_4^{(k)} b_2 v_4. \end{aligned}\right\} \tag{14}$$

When condition (B) is imposed upon (14), it follows that, at $\tau = 0$,

$$\left.\begin{aligned} x^{(0)} &= \eta_1^{(0)} + \eta_2^{(0)} + \eta_3^{(0)} + \eta_4^{(0)} = b, \\ x^{(k)} &= \eta_1^{(k)} + \eta_2^{(k)} + \eta_3^{(k)} + \eta_4^{(k)} = 0 \ (k = 1, \ldots, \infty). \end{aligned}\right\} \tag{15}$$

On substituting (12) in (11) and equating coefficients of the same powers of ϵ on both sides of the resulting equations, we obtain sets of differential equations which define the $\eta_j^{(k)}$ and $w^{(k)}$. The arbitrary constants arising at each step of the integration can be uniquely determined, as it will be shown, by the two conditions (A) and (B).

The differential equations for the terms independent of ϵ are

$$\left.\begin{aligned} \dot{\eta}_1^{(0)} - i\beta_1 \eta_1^{(0)} = 0, \quad \dot{\eta}_3^{(0)} - \beta_2 \eta_3^{(0)} = 0, \\ \dot{\eta}_2^{(0)} + i\beta_1 \eta_2^{(0)} = 0, \quad \dot{\eta}_4^{(0)} + \beta_2 \eta_4^{(0)} = 0, \\ \ddot{w}^{(0)} + W_0 w^{(0)} = 0. \end{aligned}\right\} \tag{16}$$

The general solutions of these equations are

$$\left.\begin{aligned} \eta_1^{(0)} &= a_1^{(0)} e^{i\beta_1\tau}, \quad \eta_3^{(0)} = a_3^{(0)} e^{\beta_2\tau}, \\ \eta_2^{(0)} &= a_2^{(0)} e^{-i\beta_1\tau}, \quad \eta_4^{(0)} = a_4^{(0)} e^{-\beta_2\tau}, \\ w^{(0)} &= \bar{a}_1^{(0)}\phi + \bar{a}_2^{(0)}(\chi + A\tau\phi), \end{aligned}\right\} \tag{17}$$

where the $a_1^{(0)}, \ldots, \bar{a}_2^{(0)}$ are the constants of integration. When these values of $\eta_j^{(0)}$ are substituted in (14), and condition (A) is imposed, it follows that

$$a_1^{(0)} = a_2^{(0)} = a_3^{(0)} = \bar{a}_1^{(0)} = \bar{a}_2^{(0)} = 0.$$

The remaining arbitrary constant $a_4^{(0)}$ must have the value b in order that condition (B) shall be satisfied. Hence the desired solutions for the terms independent of ϵ are

$$x^{(0)} = be^{-\beta_2\tau} u_4, \quad y^{(0)} = -bb_2 e^{-\beta_2\tau} v_4, \quad w^{(0)} = 0.$$

In determining the remaining $x^{(k)}$ and $y^{(k)}$ it will be observed that, as at this step, the coefficients of $e^{-\beta_2\tau}$ in the various $x^{(k)}$ and $y^{(k)}$ are, respectively, u_4 and $-b_2 v_4$ multiplied by the same constant factor.

From the terms in ϵ to the first degree, we obtain differential equations whose left members are the same as (16), except for the superscript 1 instead of 0. Let us denote the corresponding right members by

$$N_j^{(1)} \quad (j = 1, 2, 3, 4),$$

and $W^{(1)}$. Since X and Y in (5) contain no terms independent of ϵ, and since $w^{(0)} = 0$, it follows that

$$N_j^{(1)} = 0 \quad (j = 1, 2, 3, 4).$$

Hence the solutions for the $\eta_j^{(1)}$ are the same as $\eta_j^{(0)}$ in (17), except for the notation for the arbitrary constants, and all these constants must be zero in order that conditions (A) and (B) shall be satisfied. Therefore

$$x^{(1)} = y^{(1)} = 0.$$

There remains at this step of the integration the differential equation

$$\ddot{w}^{(1)} + W_0 w^{(1)} = W^{(1)} = b^2 e^{-2\beta_2\tau}(\gamma + \sigma), \tag{18}$$

where $(\gamma + \sigma)$, with or without subscripts or superscripts, denotes a power series in odd powers of a with sums of cosines and sines of odd multiples of τ in the coefficients. The numerical coefficients are all real, and the

highest multiple of τ in the coefficient of a^{2k+1} is $2k+1$. The complementary function of (18) is

$$w^{(1)} = D_1^{(1)}\phi + D_2^{(1)}[\chi + A\tau\phi], \tag{19}$$

where $D_1^{(1)}$ and $D_2^{(1)}$ are the constants of integration. Considering these constants as functions of τ, according to the method of the variation of parameters, we obtain

$$\dot{D}_1^{(1)} = -[\chi + A\tau\phi]\, W^{(1)}, \quad \dot{D}_2^{(1)} = \phi W^{(1)}. \tag{20}$$

When these integrations are performed and the resulting equations for $D_1^{(1)}$ and $D_2^{(1)}$ are substituted in (19), the terms of the particular integral which contain τ outside of exponential or trigonometric symbols cancel off. Thus the general solution of (18) is

$$w^{(1)} = d_1^{(1)}\phi + d_2^{(1)}[\chi + A\tau\phi] + b^2 e^{-2\beta_2\tau}(\gamma^{(1)} + \sigma^{(1)}),$$

where $d_1^{(1)}$ and $d_2^{(1)}$ are the constants of integration. From condition (A) it follows that $d_1^{(1)} = d_2^{(1)} = 0$, and the desired solution of (18) is the particular integral

$$w^{(1)} = b^2 e^{-2\beta_2\tau}(\gamma^{(1)} + \sigma^{(1)}).$$

The differential equations arising from the coefficients of ϵ^2 are the same as (16) in their left members, except for the superscript 2. Denoting the corresponding right members by

$$N_j^{(2)} \quad (j = 1, 2, 3, 4)$$

and $W^{(2)}$, we find that

$$N_1^{(2)} = b^3 e^{-3\beta_2\tau}[(C_{13}^{(2)} + S_{13}^{(2)}) + i(C_{23}^{(2)} + S_{23}^{(2)})],$$

$$N_2^{(2)} = b^3 e^{-3\beta_2\tau}[(C_{13}^{(2)} + S_{13}^{(2)}) - i(C_{23}^{(2)} + S_{23}^{(2)})],$$

$$N_3^{(2)} = b^3 e^{-3\beta_2\tau}(C_{33}^{(2)} + S_{33}^{(2)}),$$

$$N_4^{(2)} = b^3 e^{-3\beta_2\tau}(C_{43}^{(2)} + S_{43}^{(2)}),$$

$$W^{(2)} = 0.$$

The complementary functions of the differential equations at this step are the same as (17), and from condition (A) it follows that all the constants of integration must be zero, except the one associated with the complementary function $e^{-\beta_2\tau}$. Since the right members $N_1^{(2)}$, $N_2^{(2)}$, $N_3^{(2)}$, and $N_4^{(2)}$ contain no terms in $e^{i\beta_1\tau}$, $e^{-i\beta_1\tau}$, $e^{\beta_2\tau}$, and $e^{-\beta_2\tau}$, respectively, the particular integrals will yield no terms in τ explicitly, but will be of the same form

as the right members. Thus

$$\eta_1^{(2)} = b^3 e^{-3\beta_2\tau}\left[(c_{13}^{(2)}+s_{13}^{(2)})+i(c_{23}^{(2)}+s_{23}^{(2)})\right],$$

$$\eta_2^{(2)} = b^3 e^{-3\beta_2\tau}\left[(c_{13}^{(2)}+s_{13}^{(2)})-i(c_{23}^{(2)}+s_{23}^{(2)})\right],$$

$$\eta_3^{(2)} = b^3 e^{-3\beta_2\tau}(c_{33}^{(2)}+s_{33}^{(2)}),$$

$$\eta_4^{(2)} = a_4^{(2)} e^{-\beta_2\tau}+b^3 e^{-3\beta_2\tau}(c_{43}^{(2)}+s_{43}^{(2)}),$$

where $a_4^{(2)}$ is the only constant of integration which does not vanish as a consequence of condition (A). When these values are substituted in (14), we obtain

$$x^{(2)} = b^3 e^{-3\beta_2\tau}(C_3^{(2)}+S_3^{(2)})+a_4^{(2)} e^{-\beta_2\tau} u_4,$$

$$y^{(2)} = b^3 e^{-3\beta_2\tau}(\bar{C}_3^{(2)}+\bar{S}_3^{(2)})-a_4^{(2)} e^{-\beta_2\tau} b_2 v_4,$$

$$w^{(2)} = 0.$$

Since $x^{(2)}(0) = 0$, from condition (B),

$$a_4^{(2)} = -b^3 C_3^{(2)}(0).$$

The solutions at this stage may therefore be written

$$x^{(2)} = b^3\left[e^{-\beta_2\tau}(C_1^{(2)}+S_1^{(2)})+e^{-3\beta_2\tau}(C_3^{(2)}+S_3^{(2)})\right],$$

$$y^{(2)} = b^3\left[e^{-\beta_2\tau}(\bar{C}_1^{(2)}+\bar{S}_1^{(2)})+e^{-3\beta_2\tau}(\bar{C}_3^{(2)}+\bar{S}_3^{(2)})\right],$$

$$w^{(2)} = 0.$$

As in the previous step, we observe that the coefficients of $e^{-\beta_2\tau}$ in $x^{(2)}$ and $y^{(2)}$ are, respectively, u_4 and $-b_2 v_4$ multiplied by the same constant factor.

The remaining steps of the integration can be carried on in precisely the same way. The solutions already obtained are sufficient to reveal the general nature of the various $x^{(n)}$, $y^{(n)}$, and $w^{(n)}$, and, by an induction to the general term, it can readily be shown that

$$x^{(2k)} = b^{2k+1}\sum_{j=0}^{k} e^{-(2j+1)\beta_2\tau}(C_{2j+1}^{(2k)}+S_{2j+1}^{(2k)}),$$

$$y^{(2k)} = b^{2k+1}\sum_{j=0}^{k} e^{-(2j+1)\beta_2\tau}(\bar{C}_{2j+1}^{(2k)}+\bar{S}_{2j+1}^{(2k)}),$$

$$w^{(2k)} = 0;$$

and $$x^{(2k+1)} = y^{(2k+1)} = 0,$$

$$w^{(2k+1)} = b^{2k+2} \sum_{j=1}^{k} e^{-2j\beta_2\tau} (\gamma_{2j}^{(2k+1)} + \sigma_{2j}^{(2k+1)}).$$

The terms of $x^{(2k)}$ and $y^{(2k)}$ which carry $e^{-\beta_2\tau}$ as a factor arise only from the complementary function. They are

$$x^{(2k)} = a_4^{(2k)} e^{-\beta_2\tau} u_4, \qquad y^{(2k)} = -a_4^{(2k)} e^{-\beta_2\tau} b_2 v_4,$$

where $a_4^{(2k)}$ is the constant of integration. Hence the coefficients of $e^{-\beta_2\tau}$ in the various $x^{(n)}$ and $y^{(n)}$ are, respectively, u_4 and $-b_2 v_4$ multiplied by a constant which varies only with n.

On substituting these values of $x^{(n)}$, $y^{(n)}$, and $w^{(n)}$ in (13), and the second equation of (12), and restoring the strokes which were dropped after equations (4), we obtain

$$\left.\begin{aligned} \bar{x} &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \epsilon^{2k} b^{2k+1} e^{-(2j+1)\beta_2\tau} (C_{2j+1}^{(2k)} + S_{2j+1}^{(2k)}), \\ \bar{y} &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \epsilon^{2k} b^{2k+1} e^{-(2j+1)\beta_2\tau} (\bar{C}_{2j+1}^{(2k)} + \bar{S}_{2j+1}^{(2k)}), \\ \bar{w} &= \sum_{k=0}^{\infty} \sum_{j=1}^{k} \epsilon^{2k+1} b^{2k+2} e^{-2j\beta_2\tau} (\gamma_{2j}^{(2k+1)} + \sigma_{2j}^{2k+1)}). \end{aligned}\right\} \tag{21}$$

Since we must multiply these values of $\bar{x}$, $\bar{y}$, and $\bar{w}$ by ϵ in order to obtain x, y, and w, it is apparent that ϵ and b occur in each term of (21) as a factor and to the same power. Hence we may put $b = 1$ without loss of generality. Then, on changing the superscripts of (21) to correspond with the powers of ϵ, we obtain the asymptotic solutions,

$$\left.\begin{aligned} x &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \epsilon^{2k+1} e^{-(2j+1)\beta_2\tau} (C_{2j+1}^{(2k+1)} + S_{2j+1}^{(2k+1)}), \\ y &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \epsilon^{2k+1} e^{-(2j+1)\beta_2\tau} (\bar{C}_{2j+1}^{(2k+1)} + \bar{S}_{2j+1}^{(2k+1)}), \\ z &= \psi + w = \psi + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \epsilon^{2k} e^{-2j\beta_2\tau} [\gamma_{2j}^{(2k)} + \sigma_{2j}^{(2k)}], \end{aligned}\right\} \tag{22}$$

where the symbolic expressions in the brackets ( ) denote power series in a^2 with sums of cosines and sines of even multiples of τ in the coefficients, and the terms $[\gamma_{2j}^{(2k)} + \sigma_{2j}^{(2k)}]$ denote power series in odd powers of a with sums of cosines and sines of odd multiples of τ in the coefficients.

As τ approaches $+\infty$, the most important terms in (22) are those containing $e^{-\beta_2\tau}$ to the first degree. It was pointed out that the coefficients

of $e^{-\beta_2\tau}$ in $x^{(n)}$ and $y^{(n)}$, at each step of the integration, are $a_4^{(n)}u_4$ and $-a_4^{(n)}b_2v_4$ respectively, where $a_4^{(n)}$ is a constant. Hence, in so far as the terms in $e^{-\beta_2\tau}$ are concerned, we have

$$\left.\begin{aligned} x &= \quad e^{-\beta_2\tau}u_4[\epsilon+p_3\epsilon^3+\ldots+p_{2n+1}\epsilon^{2n+1}+\ldots], \\ y &= -e^{-\beta_2\tau}b_2v_4[\epsilon+p_3\epsilon^3+\ldots+p_{2n+1}\epsilon^{2n+1}+\ldots], \end{aligned}\right\} \tag{23}$$

where $p_3, \ldots, p_{2n+1}, \ldots$ are known constants.

The asymptotic solutions which approach zero as τ and therefore t approach $-\infty$ can be constructed in an entirely similar manner, but it is not necessary to repeat this construction, as the following consideration will show that the latter solutions can be obtained directly from the former.

Consider the differential equations (5), and let their solutions (21) be denoted by

$$x = f_1(\tau), \quad y = f_2(\tau), \quad w = f_3(\tau). \tag{24}$$

The coefficients of the dependent variables and their derivatives in the left members of (3), and also the right members X and Y, are even functions of τ, but W is an odd function of τ. Further, X is even in y, w, and τ, considered singly or together; Y is odd in y but even in w and τ, considered together; and finally, W is even in y, but odd in w and τ, considered together. Therefore, if we make in (5) the substitutions

$$x = x, \quad y = -p, \quad w = -q, \quad \tau = -\tau_1, \tag{25}$$

the resulting differential equations in the new variables x, p, q, and τ_1 are identically the same as the former, equations (5), are in x, y, w, and τ. With the same initial conditions in the new variables, viz.,

$$x(0) = x(0), \quad p(0) = y(0), \quad q(0) = w(0),$$

$$\left(\frac{dx}{d\tau_1}\right)_0 = \left(\frac{dx}{d\tau}\right)_0, \quad \left(\frac{dp}{d\tau_1}\right)_0 = \left(\frac{dy}{d\tau}\right)_0, \quad \left(\frac{dq}{d\tau_1}\right)_0 = \left(\frac{dw}{d\tau}\right)_0,$$

the solutions of the differential equations in x, p, q, and τ_1 are

$$x = f_1(\tau_1), \quad p = f_2(\tau_1), \quad q = f_3(\tau_1),$$

where f_1, f_2, and f_3 denote the same functions as in (24). On restoring the former variables by (25), we have

$$x = f_1(-\tau), \quad y = -f_2(-\tau), \quad w = -f_3(-\tau),$$

as solutions of (5). That is, the asymptotic solutions which approach zero as τ approaches $-\infty$, may be obtained from (21) by changing the signs of y, w, and τ in (21). Then on restoring the strokes, multiplying

by ϵ, and putting $b = 1$, as in the former solutions, we obtain

$$\left.\begin{aligned} x &= +\sum_{k=0}^{\infty}\sum_{j=0}^{k} \epsilon^{2k+1} e^{(2j+1)\beta_2\tau}(C_{2j+1}^{(2k+1)} - S_{2j+1}^{(2k+1)}), \\ y &= -\sum_{k=0}^{\infty}\sum_{j=0}^{k} \epsilon^{2k+1} e^{(2j+1)\beta_2\tau}(\bar{C}_{2j+1}^{(2k+1)} - \bar{S}_{2j+1}^{(2k+1)}), \\ z &= \psi - w = \psi - \sum_{k=1}^{\infty}\sum_{j=1}^{k} \epsilon^{2k} e^{2j\beta_2\tau}[\gamma_{2j}^{(2k)} - \sigma_{2j}^{(2k)}]. \end{aligned}\right\} \tag{26}$$

These solutions differ from (22) only in the signs indicated.

For sufficiently large negative values of τ, the series for x and y will approximate to

$$\left.\begin{aligned} x &= e^{\beta_2\tau} u_4 [\epsilon + p_3\epsilon^3 + \ldots + p_{2n+1}\epsilon^{2n+1} + \ldots], \\ y &= e^{\beta_2\tau} b_2 v_4 [\epsilon + p_3\epsilon^3 + \ldots + p_{2n+1}\epsilon^{2n+1} + \ldots], \end{aligned}\right\} \tag{27}$$

where $p_3, \ldots, p_{2n+1}, \ldots$ are the same constants as in (23).

7. *The Convergence of the Asymptotic Solutions.*

Poincaré has shown* that asymptotic solutions similar to (22) will converge for sufficiently large values of τ, provided that the denominators which are introduced in the construction of the solutions are different from zero. In case the denominators are zero, the corresponding particular integrals will not be $e^{-\nu\beta_2\tau}$, ν an integer, multiplied by a periodic function, but will contain the factor τ explicitly, and hence these terms would approach infinity with τ.

In the first four equations of (11) the denominators have the form

$$D_1 = i\lambda - \nu\beta_2 - i\beta_1, \qquad D_3 = i\lambda - \nu\beta_2 - \beta_2,$$
$$D_2 = i\lambda - \nu\beta_2 + i\beta_1, \qquad D_4 = i\lambda - \nu\beta_2 + \beta_2,$$

where λ is a positive even integer or zero, and ν is an odd positive integer. The first two denominators can vanish only if $\lambda = \pm\beta_1$ and $\nu = 0$, but this is impossible, since $\nu \neq 0$, and β_1 is irrational in general. The value of D_3 may be zero for $\lambda = 0$ and $\nu = -1$. Now ν would have the value -1, provided the right members of the differential equations defining the various $\eta_3^{(k)}$ contained terms in $e^{\beta_2\tau}$. Such terms appear in the complementary functions of these differential equations, but they are made to disappear by choosing zero values for the corresponding constants of

* *Mécanique céleste*, Vol. 1, Chap. VIII, § 105.

integration. Hence terms in $e^{\beta_2\tau}$ do not appear in the right members of these differential equations, and D_3 cannot vanish.

The denominator D_4 will vanish for $\lambda = 0$, $\nu = 1$. This condition arises when the right members of the various differential equations

$$\dot{\eta}_4^{(k)} + \beta_2 \eta_4^{(k)} = N_4^{(k)} \quad (k = 1, \ldots, \infty),$$

contain the term $e^{-\beta_2\tau}$ multiplied by constants. As x and y do not occur linearly in X and Y of (5), and since $w^{(0)} = 0$ and the remaining $w^{(k)}$ are either zero or contain $e^{-2j\beta_2\tau}$ $(j > 0)$ as a factor, it follows that the term $e^{-\beta_2\tau}$ does not appear in any of the $N_4^{(k)}$. Consequently the denominator D_4 cannot vanish.

The terms in τ which occur explicitly in the integration of the various equations in $w^{(k)}$ either cancel off or are made to vanish as a consequence of condition (A), by choosing zero values for the constants of integration.

Thus no terms in τ occur explicitly in the solutions (22), and, according to Poincaré's theorem, these solutions will converge for sufficiently large values of τ.

On substituting

$$\tau = \frac{t - t_0}{\left[\frac{1}{8}(1+\delta)\right]^{\frac{1}{2}}}, \qquad \delta = \tfrac{9}{2}a^2 - \tfrac{141}{32}a^4 + \tfrac{35}{2}a^6 + \ldots,$$

in (22), we obtain the solutions in terms of the time t, and they will converge as t approaches $+\infty$. Similarly it may be established that the asymptotic solutions (26) constructed in terms of $e^{\beta_2\tau}$ will converge as τ and therefore t approach $-\infty$.

8. *Geometrical Considerations.*

The solutions (22) and (26) contain the two arbitrary constants, t_0 and ϵ. The initial time t_0, can be chosen to be zero without loss of geometric generality. From the way in which the condition (B) was chosen, it follows that the parameter ϵ denotes the x-component of the initial displacement from the linear orbit

$$x \equiv y \equiv 0, \qquad z = \psi.$$

By a suitable change in condition (B), the constant ϵ may be made to represent the y- or z-component of the initial displacement from the linear orbit, or the x-, y-, or z-component of the initial projection from this linear orbit.

Let us consider the projection on the rotating xy-plane of the orbit or orbits represented by (22). As x and y approach the origin, that is, for

sufficiently large values of τ, their values are determined by (23). Now for $a = 0$,

$$u_4 = v_4 = 1, \qquad b_2 = -0{\cdot}35\ldots,$$

and for a different from zero, but sufficiently small numerically, the above signs of u_4, v_4, and b_2 will not change. Hence both x and y have the same sign as ϵ in the neighbourhood of the origin. When ϵ is positive, therefore, the projection of the orbit in the xy-plane leaves the origin in the first quadrant of the rotating plane; when ϵ is negative, this curve leaves the origin in the third quadrant. Consequently, equations (22) represent two orbits, one orbit for positive values of ϵ and another for negative values of ϵ.

Since x and y are odd in ϵ and w is even in ϵ, these two orbits are not geometrically distinct, as the one is the reflection of the other, first with respect to the x-axis and then with respect to the y-axis. This symmetry would be expected, since the finite masses are equal.

Similarly for the equations (26), the signs of x and y in the neighbourhood of the origin are given by (27). For sufficiently small values of $|a|$, x has the same sign as ϵ, but y has the opposite sign. Hence, when ϵ is positive, the projection of the orbit in the xy-plane leaves the origin in the fourth quadrant of the rotating plane, and, when ϵ is negative, this curve leaves the origin in the second quadrant. Therefore, equations (26) likewise represent two orbits, but, as in the preceding paragraph, they have the same shape and are only differently orientated.

The directions in which the projections of the various orbits on the xy-plane leave the origin are given by dy/dx for $\tau = \pm\infty$. Considering equations (23) first, we have the slope given by

$$\frac{dy}{dx} = \lim_{\tau=+\infty} \frac{dy}{d\tau}\Big/\frac{dx}{d\tau} = \lim_{\tau=+\infty} \frac{b_2 e^{-\beta_2\tau}(\beta_2 v_4 + \dot{v}_4)(\epsilon + p_3\epsilon^3 + \ldots)}{-e^{-\beta_2\tau}(\beta_2 u_4 - \dot{u}_4)(\epsilon + p_3\epsilon^3 + \ldots)}$$

$$= -b_2 \lim_{\tau=+\infty} \frac{\beta_2 v_4 + v_4}{\beta_2 u_4 - \dot{u}_4}.$$

Similarly for the equations (27),

$$\frac{dy}{dx} = b_2 \lim_{\tau=-\infty} \frac{\beta_2 v_4 + \dot{v}_4}{\beta_2 u_4 - \dot{u}_4}.$$

Obviously these directions are independent of ϵ. They are, moreover, indeterminate, as u_4, v_4, $\dot{u}_4$, and $\dot{v}_4$ contain sums of sines and cosines of multiples of τ in the coefficients of the various powers of a^2, and these functions do not approach definite limiting values as their arguments approach infinity.

9. *Numerical Example.*

To illustrate the nature of the above asymptotic orbits, let us consider a particular case when the scale factors of the periodic and asymptotic orbits, viz., a and ϵ respectively, are both equal to $\frac{1}{2}$.

If we neglect the terms beyond a^2 or ϵ^2, then the desired solutions of (5), when multiplied by ϵ to return to the variables x, y, and z, are found to be

$$x = e^{-\beta_2\tau}\epsilon u_4, \qquad y = -b_2 e^{-\beta_2\tau}\epsilon v_4,$$

$$w = -\frac{23{\cdot}28a\epsilon^2}{4(\beta_2^2+1)}\, e^{-2\beta_2\tau}\left(\sin\tau + \frac{1}{\beta_2}\cos\tau\right),$$

where

$$u_4 = 1+a^2(1{\cdot}09-1{\cdot}09\cos 2\tau-1{\cdot}48\sin 2\tau)+\ldots,$$

$$-b_2v_4 = 0{\cdot}35+a^2(0{\cdot}39-0{\cdot}52\cos 2\tau+0{\cdot}13\sin 2\tau)+\ldots,$$

$$\beta_2 = 1{\cdot}34-1{\cdot}34a^2+\ldots.$$

On substituting $a = \epsilon = \frac{1}{2}$ in the preceding equations, we obtain

$$\left.\begin{aligned} x &= e^{-1{\cdot}005\tau}(0{\cdot}636-0{\cdot}136\cos 2\tau-0{\cdot}185\sin 2\tau)+\ldots,\\ y &= e^{-1{\cdot}005\tau}(0{\cdot}224-0{\cdot}065\cos 2\tau+0{\cdot}016\sin 2\tau)+\ldots,\\ w &= -e^{-2{\cdot}01\tau}\,0{\cdot}036\,(\sin\tau+\cos\tau)+\ldots,\\ \beta_2 &= 1{\cdot}005. \end{aligned}\right\} \qquad (28)$$

Numerical values of the coordinates, with respect to both the rotating and the fixed axes, are given in the following table.

τ	x	y	w	ξ	η	$\zeta = z = \psi + w$
·0	·500	·160	−·036	·500	·160	−·036
·1	·421	·146	−·031	·420	·148	+·019
·2	·359	·140	−·027	·358	·143	+·073
·3	·310	·132	−·025	·308	·136	+·125
·4	·273	·128	−·019	·271	·133	+·171
·5	·246	·123	−·018	·243	·128	+·222
1	·191	·097	−·007	·185	·104	+·413
1·5	·165	·065	−·002	·161	·074	+·498
2	·129	·034	·000	·125	·045	+·450
2·5	·063	·016	·000	·063	·023	+·299
3	·027	·008	·000	·026	·011	+·070
3·5	·012	·006	·000	·011	·008	−·175
4	·008	·004	·000	·007	·005	−·378

The projections of the asymptotic orbit on the fixed coordinate planes

are given in the accompanying figures. The periodic orbit is a straight

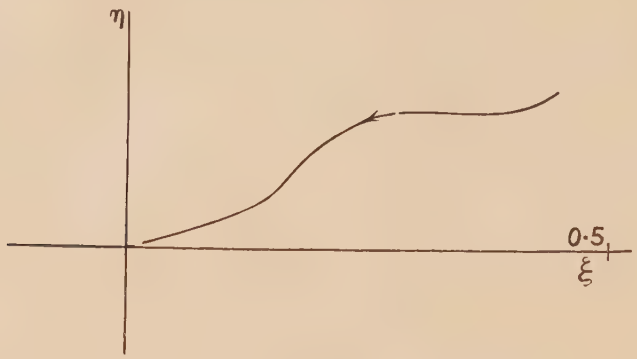

FIG. 1.

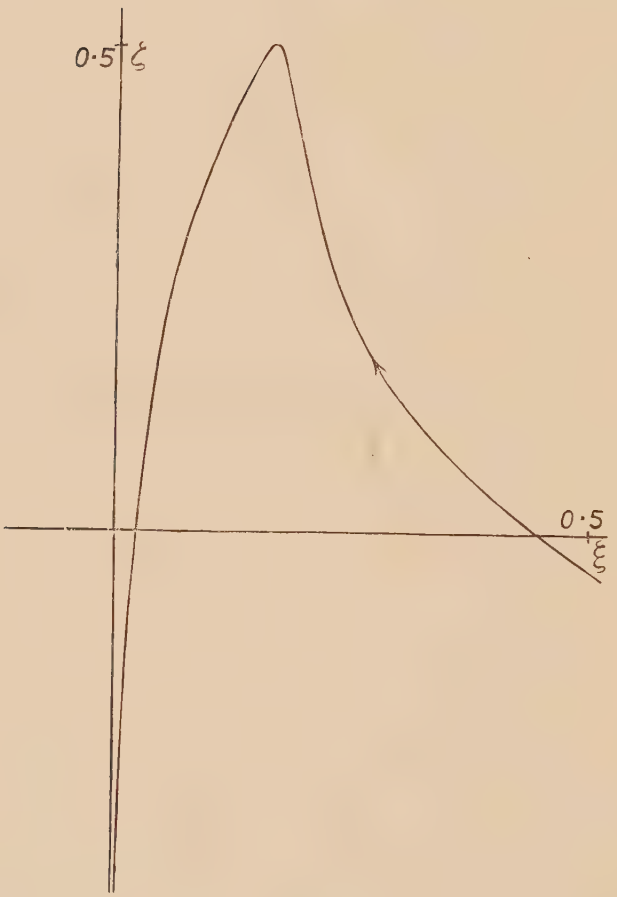

FIG. 2.

line coinciding with the ξ-axis, and for $a = \frac{1}{2}$ it extends a half-unit, approximately, above and below the $\xi\eta$-plane. The values of x, y, and w denote the amount by which the asymptotic orbit deviates from the periodic orbit. The diagrams are drawn for τ varying from 0 to 4.

FIG. 3.

For $\tau > 4$, the deviation from the origin or from the ξ-axis is so slight that it is difficult to draw the curves according to the scale which has been used. The remaining portion of the curve in Fig. 1 consists of a spiral about the origin. Since x and y in (28) are both positive for all values of τ, the spiral winds about the origin in the counter-clockwise direction as τ increases. In Figs. 2 and 3 the curves oscillate near the ξ-axis and between the parallel lines at the distance $\frac{1}{2}$, approximately, on either side of the ξ and η axes respectively. These curves cross the

ξ-axis, but their greatest deviation from the ξ-axis constantly diminishes as τ increases.

The period for a complete revolution about the origin in Fig. 1 is 2π in t, while the period for a complete oscillation in Figs. 2 and 3 is 2π in τ. Since

$$t-t_0 = \frac{\tau}{[\frac{1}{8}(1+\delta)]^{\frac{1}{2}}} = \frac{\tau}{\cdot 481},$$

for $a = \frac{1}{2}$, the period of revolution is to the period of oscillation as 0·481 is to 1. After the curves have crossed the ξ-axis in Figs. 2 and 3 they will cross it again in a half period of revolution, and in that interval they will have made about one complete oscillation.

ASYMPTOTIC FORMULÆ IN COMBINATORY ANALYSIS

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1.

INTRODUCTION AND SUMMARY OF RESULTS.

1 . 1. The present paper is the outcome of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory of analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represent certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$(1 . 11) \qquad f(x) = 1 + \sum_1^\infty p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.\dagger$$

If we write

$$(1 . 12) \qquad x = q^2 = e^{2\pi i \tau},$$

where the imaginary part of τ is positive, we see that $f(x)$ is substantially the reciprocal of the modular function called by Tannery and Molk‡ $h(\tau)$; that, in fact,

$$(1 . 13) \qquad h(\tau) = q^{\frac{1}{12}} q_0 = q^{\frac{1}{12}} \prod_1^\infty (1-q^{2n}) = \frac{x^{\frac{1}{24}}}{f(x)}.$$

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* A short abstract of the contents of part of this paper appeared under the title "Une formule asymptotique pour le nombre des partitions de n", in the *Comptes rendus*, January 2nd, 1917.

† P. A. MacMahon, *Combinatory Analysis*, Vol. 2, 1916, p. 1.

‡ J. Tannery and J. Molk, *Fonctions elliptiques*, Vol. 2, 1896, pp. 31 *et seq.* We shall follow the notation of this work whenever we have to quote formulæ from the theory of elliptic functions.

The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities—many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formulæ, one may fairly say, there are none.* So true is this, in fact, that we have been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of $p(n)$.

1.2. The function $p(n)$ may, of course, be expressed in the form of an integral

$$p(n) = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x^{n+1}}\,dx, \tag{1.21}$$

by means of Cauchy's theorem, the path Γ enclosing the origin and lying entirely inside the unit circle. The idea which dominates this paper is that of obtaining asymptotic formulæ for $p(n)$ by a detailed study of the

* We should mention one exception to this statement, to which our attention was called by Major MacMahon. The number of partitions of n *into parts none of which exceed* r is the coefficient $p_r(n)$ in the series

$$1+\sum_1^\infty p_r(n)\,x^n = \frac{1}{(1-x)(1-x^2)\ldots(1-x^r)}.$$

This function has been studied in much detail, for various special values of r, by Cayley, Sylvester, and Glaisher: we may refer in particular to J. J. Sylvester, "On a discovery in the theory of partitions", *Quarterly Journal*, Vol. 1, 1857, pp. 81–85, and "On the partition of numbers", *ibid.*, pp. 141–152 (Sylvester's *Works*, Vol. 2, pp. 86–89 and 90–99); J. W. L. Glaisher, "On the number of partitions of a number into a given number of parts", *Quarterly Journal*, Vol. 40, 1909, pp. 57–143; "Formulæ for partitions into given elements, derived from Sylvester's Theorem", *ibid.*, pp. 275–348; "Formulæ for the number of partitions of a number into the elements 1, 2, 3, ..., n up to $n=9$", *ibid.*, Vol. 41, 1910, pp. 94–112: and further references will be found in MacMahon, *l.c.*, pp. 59–71, and E. Netto, *Lehrbuch der Combinatorik*, 1901, pp. 146–158. Thus, for example, the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

is

$$p_3(n) = \frac{1}{12}(n+3)^2 - \frac{7}{72} + \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3};$$

as is easily found by separating the function into partial fractions. This function may also be expressed in the orms

$$\tfrac{1}{12}(n+3)^2 + (\tfrac{1}{2}\cos\tfrac{1}{2}\pi n)^2 - (\tfrac{2}{3}\sin\tfrac{1}{3}\pi n)^2,$$

$$1+[\tfrac{1}{12}n(n+6)],\quad \left\{\tfrac{1}{12}(n+3)^2\right\},$$

where $[n]$ and $\{n\}$ denote the greatest integer contained in n and the integer nearest to n. These formulæ do, of course, furnish incidentally asymptotic formulæ for the functions in question. But they are, from this point of view, of a very trivial character: the interest which they possess is algebraical.

integral (1 . 21). This idea is an extremely obvious one; it is the idea which has dominated nine-tenths of modern research in the analytic theory of numbers: and it may seem very strange that it should never have been applied to this particular problem before. Of this there are no doubt two explanations. The first is that the theory of partitions has received its most important developments, since its foundation by Euler, at the hands of a series of mathematicians whose interests have lain primarily in algebra. The second and more fundamental reason is to be found in the extreme complexity of the behaviour of the generating function $f(x)$ near a point of the unit circle.

It is instructive to contrast this problem with the corresponding problems which arise for the arithmetical functions $\pi(n)$, $\vartheta(n)$, $\psi(n)$, $\mu(n)$, $d(n)$, ... which have their genesis in Riemann's Zeta-function and the functions allied to it. In the latter problems we are dealing with functions defined by Dirichlet's series. The study of such functions presents difficulties far more fundamental than any which confront us in the theory of the modular functions. These difficulties, however, relate to the distribution of the zeros of the functions and their general behaviour at infinity: no difficulties whatever are occasioned by the crude singularities of the functions in the finite part of the plane. The single finite singularity of $\zeta(s)$, for example, the pole at $s = 1$, is a singularity of the simplest possible character. It is this pole which gives rise to the *dominant* terms in the asymptotic formulæ for the arithmetical functions associated with $\zeta(s)$. To prove such a formula rigorously is often exceedingly difficult; to determine precisely the order of the error which it involves is in many cases a problem which still defies the utmost resources of analysis. But to write down the dominant terms involves, as a rule, no difficulty more formidable than that of deforming a path of integration over a pole of the subject of integration and calculating the corresponding residue.

In the theory of partitions, on the other hand, we are dealing with functions which do not exist at all outside the unit circle. Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a "dominant term."

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulæ of the theory of the linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behaviour of $f(x)$ near any

assigned point of the unit circle.* It is to an appropriate use of these formulæ that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

1 . 3. It is very important, in dealing with such a problem as this, to distinguish clearly the various stages to which we can progress by arguments of a progressively "deeper" and less elementary character. The earlier results are naturally (so far as the particular problem is concerned) superseded by the later. But the more elementary methods are likely to be applicable to other problems in which the more subtle analysis is impracticable.

We have attacked this particular problem by a considerable number of different methods, and cannot profess to have reached any very precise conclusions as to the possibilities of each. A detailed comparison of the results to which they lead would moreover expand this paper to a quite unreasonable length. But we have thought it worth while to include a short account of two of them. The first is quite elementary ; it depends only on Euler's identity

$$(1 . 31) \qquad \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = 1+\frac{x}{(1-x)^2}+\frac{x^4}{(1-x)^2(1-x^2)^2}+\ldots,$$

—an identity capable of wide generalisation—and on elementary algebraical reasoning. By these means we show, in section 2, that

$$(1 . 32) \qquad e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}},$$

where A and B are positive constants, for all sufficiently large values of n.

It follows that

$$(1 . 33) \qquad A\sqrt{n} < \log p(n) < B\sqrt{n} ;$$

and the next question which arises is the question whether a constant C exists such that

$$(1 . 34) \qquad \log p(n) \sim C\sqrt{n}.$$

We prove that this is so in section 3. Our proof is still, in a sense, "elementary". It does not appeal to the theory of analytic functions, depending only on a general arithmetic theorem concerning infinite series ;

* See G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation (II: The trigonometrical series associated with the elliptic Theta-functions)", *Acta Mathematica*, Vol. 37, 1914, pp. 193–238, for applications of the formulæ to different but not unrelated problems.

but this theorem is of the difficult and delicate type which Messrs. Hardy and Littlewood have called "Tauberian". The actual theorem required was proved by us in a paper recently printed in these *Proceedings**. It shows that

$$C = \frac{2\pi}{\sqrt{6}}; \tag{1.35}$$

in other words that

$$p(n) = \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}(1+\epsilon)\right\}, \tag{1.36}$$

where ϵ is small when n is large. This method is one of very wide application. It may be used, for example, to prove that, if $p^{(s)}(n)$ denotes the number of partitions of n into perfect s-th powers, then

$$\log p^{(s)}(n) \sim (s+1)\left\{\frac{1}{s}\,\Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right)\right\}^{s/(s+1)} n^{1/(s+1)}.$$

It is certainly possible to obtain, by means of arguments of this general character, information about $p(n)$ more precise than that furnished by the formula (1.36). And it is equally possible to prove (1.36) by reasoning of a more elementary, though more special, character: we have a proof, for example, based on the identity

$$np(n) = \sum_{\nu=1}^{n} \sigma(\nu)\, p(n-\nu),$$

where $\sigma(\nu)$ is the sum of the divisors of ν, and a process of induction. But we are at present unable to obtain, by any method which does not depend upon Cauchy's theorem, a result as precise as that which we state in the next paragraph, a result, that is to say, which is "vraiment asymptotique".

1.4. Our next step was to replace (1.36) by the much more precise formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}.\dagger \tag{1.41}$$

The proof of this formula appears to necessitate the use of much more

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* G. H. Hardy and S. Ramanujan, "Asymptotic formulæ concerning the distribution of integers of various types", *Proc. London Math. Soc.*, Ser. 2, Vol. 16, 1917, pp. 112–132.

† In our note in the *Comptes rendus* $4n\sqrt{3}$ is misprinted as $4\pi\sqrt{3}$.

powerful machinery, Cauchy's integral (1 . 21) and the functional relation

$$(1\,.\,42) \qquad f(x) = \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}} \sqrt{\left(\log \frac{1}{x}\right)} \exp\left\{\frac{\pi^2}{6\log(1/x)}\right\} f(x'),$$

where

$$(1\,.\,43) \qquad x' = \exp\left\{-\frac{4\pi^2}{\log(1/x)}\right\}.$$

This formula is merely a statement in a different notation of the relation between $h(\tau)$ and $h(\mathrm{T})$, where

$$\mathrm{T} = \frac{c+d\tau}{a+b\tau}, \quad a=d=0, \quad b=1, \quad c=-1;$$

viz.

$$h(\tau) = \sqrt{\left(\frac{i}{\tau}\right)}\, h(\mathrm{T}).*$$

It is interesting to observe the correspondence between (1 . 41) and the results of numerical computation. Numerical data furnished to us by Major MacMahon gave the following results: we denote the right-hand side of (1 . 41) by $\varpi(n)$.

n	$p(n)$	$\varpi(n)$	ϖ/p
10	42	48·104	1·145
20	627	692·385	1·104
50	204226	217590·499	1·065
80	15796476	16606781·567	1·051

It will be observed that the progress of ϖ/p towards its limit unity is not very rapid, and that $\varpi - p$ is always positive and appears to tend rapidly to infinity.

1 . 5. In order to obtain more satisfactory results it is necessary to construct some auxiliary function $F(x)$ which is regular at all points of the unit circle save $x = 1$, and has there a singularity of a type as near as possible to that of the singularity of $f(x)$. We may then hope to ob-

* Tannery and Molk, *l.c.*, p. 265 (Table XLV, 5).

tain a much more precise approximation by applying Cauchy's theorem to $f-F$ instead of to F. For although every point of the circle is a singular point of f, $x=1$ is, to put it roughly, much the *heaviest* singularity. When $x\to 1$ by real values, $f(x)$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6(1-x)}\right\};$$

when

$$x = re^{2p\pi i/q},$$

p and q being co-prime integers, and $r\to 1$, $|f(x)|$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6q^2(1-r)}\right\};$$

while, if

$$x = re^{2\theta\pi i},$$

where θ is irrational, $|f(x)|$ can become infinite at most like an exponential of the type

$$\exp\left\{o\left(\frac{1}{1-r}\right)\right\}.*$$

The function required is

$$F(x) = \frac{1}{\pi\sqrt{2}}\sum_1^\infty \psi(n)\,x^n, \tag{1.51}$$

where

$$\psi(n) = \frac{d}{dn}\left\{\frac{\cosh C\lambda_n - 1}{\lambda_n}\right\}, \tag{1.52}$$

$$C = 2\pi/\sqrt{6} = \pi\sqrt{(\tfrac{2}{3})}, \qquad \lambda_n = \sqrt{(n-\tfrac{1}{24})}. \tag{1.53}$$

This function may be transformed into an integral by means of a general formula given by Lindelöf† ; and it is then easy to prove that the "principal branch" of $F(x)$ is regular all over the plane except at $x=1$‡ ;

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* The statements concerning the "rational" points are corollaries of the formulæ of the transformation theory, and proofs of them are contained in the body of the paper. The proposition concerning "irrational" points may be proved by arguments similar to those used by Hardy and Littlewood in their memoir already quoted. It is not needed for our present purpose. As a matter of fact it is *generally* true that $f(x)\to 0$ when θ is irrational, and very nearly as rapidly as $\sqrt[4]{(1-r)}$. It is in reality owing to this that our final method is so successful.

† E. Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions* (Gauthier-Villars, Collection Borel, 1905), p. 111.

‡ We speak, of course, of the principal branch of the function, viz. that represented by the series (1.51) when x is small. The other branches are singular at the origin.

and that

$$F(x)-\chi(x),$$

where

$$\chi(x)=\frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\left[\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\}-1\right] \tag{1.54}$$

is regular for $x=1$. If we compare (1.42) and (1.54), and observe that $f(x')$ tends to unity with extreme rapidity when x tends to 1 along any regular path which does not touch the circle of convergence, we can see at once the very close similarity between the behaviour of f and F inside the unit circle and in the neighbourhood of $x=1$.

It should be observed that the term -1 in (1.52) and (1.54) is—so far as our present assertions are concerned—otiose: all that we have said remains true if it is omitted; the resemblance between the singularities of f and F becomes indeed even closer. The term is inserted merely in order to facilitate some of our preliminary analysis, and will prove to be without influence on the final result.

Applying Cauchy's theorem to $f-F$, we obtain

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+O(e^{D\sqrt{n}}), \tag{1.55}$$

where D is any number greater than

$$\tfrac{1}{2}C=\tfrac{1}{2}\pi\sqrt{(\tfrac{2}{3})}.$$

1.6. The formula (1.55) is an asymptotic formula of a type far more precise than that of (1.41). The error term is, however, of an exponential type, and may be expected ultimately to increase with very great rapidity. It was therefore with considerable surprise that we found what exceedingly good results the formula gives for fairly large values of n. For $n=61, 62, 63$ it gives*

1121538·972, 1300121·359, 1505535·606,

while the correct values are

1121505, 1300156, 1505499.

The errors 33·972, −34·641, 36·606

are relatively very small, and alternate in sign.

The next step is naturally to direct our attention to the singular

* In the *Comptes rendus* we misstated the second number as 1300111.

point of $f(x)$ next in importance after that at $x = 1$, viz., that at $x = -1$; and to subtract from $f(x)$ a second auxiliary function, related to this point as $F(x)$ is to $x = 1$. No new difficulty of principle is involved, and we find that

$$(1.61) \qquad p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi} \frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}),$$

where D is now any number greater than $\frac{1}{3}C$. It now becomes obvious why our earlier approximation gave errors alternately of excess and of defect.

It is obvious that this process may be repeated indefinitely. The singularities next in importance are those at $x = e^{\frac{2}{3}\pi i}$ and $x = e^{\frac{4}{3}\pi i}$; the next those at $x = i$ and $x = -i$; and so on. The next two terms in the approximate formula are found to be

$$\frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{18}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)$$

and

$$\frac{\sqrt{2}}{\pi} \cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{8}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{4}C\lambda_n}}{\lambda_n}\right).$$

As we proceed further, the complexity of the calculations increases. The auxiliary function associated with the point $x = e^{2p\pi i/q}$ involves a certain $24q$-th root of unity, connected with the linear transformation which must be used in order to elucidate the behaviour of $f(x)$ near the point; and the explicit expression of this root in terms of p and q, though known, is somewhat complex. But it is plain that, by taking a sufficient number of terms, we can find a formula in which the error is

$$O(e^{C\lambda_n/\nu}),$$

where ν is a fixed but arbitrarily large integer.

1.7. A final question remains. We have still the recourse of making ν a function of n, that is to say of making the number of terms in our approximate formula itself a function of n. In this way we may reasonably hope, at any rate, to find a formula in which the error is of order less than that of any exponential of the type e^{an}; of the order of a power of n, for example, or even bounded.

When, however, we proceeded to test this hypothesis by means of the numerical data most kindly provided for us by Major MacMahon, we found a correspondence between the real and the approximate values of such astonishing accuracy as to lead us to hope for even more. Taking

$n = 100$, we found that the first six terms of our formula gave

$$\begin{array}{r} 190568944{\cdot}783 \\ +348{\cdot}872 \\ -2{\cdot}598 \\ +{\cdot}685 \\ +{\cdot}318 \\ -{\cdot}064 \\ \hline 190569291{\cdot}996\,, \end{array}$$

while $$p(100) = 190569292\,;$$

so that the error after six terms is only ·004. We then proceeded to calculate p (200), and found

$$\begin{array}{r} 3,\,972,\,998,\,993,\,185{\cdot}896 \\ +36,\,282{\cdot}978 \\ -87{\cdot}555 \\ +\ 5{\cdot}147 \\ +\ 1{\cdot}424 \\ +\ 0{\cdot}071 \\ +\ 0{\cdot}000* \\ +\ 0{\cdot}043 \\ \hline 3,\,972,\,999,\,029,\,388{\cdot}004\,, \end{array}$$

and Major MacMahon's subsequent calculations showed that p (200) is, in fact,

$$3,\,972,\,999,\,029,\,388.$$

These results suggest very forcibly that it is possible to obtain a formula for $p(n)$, which not only exhibits its order of magnitude and structure, but may be used to calculate its *exact* value for any value of n. That this is in fact so is shown by the following theorem.

Statement of the main theorem.

THEOREM.—*Suppose that*

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}}\,\frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right), \tag{1.71}$$

* This term vanishes identically.

where C and λ_n are defined by the equations (1 . 53), *for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula*

(1 . 721)

$$\omega_{p,q} = \left(\frac{-q}{p}\right) \exp\left[-\left\{\tfrac{1}{4}(2-pq-p)+\tfrac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

(1 . 722) $$\omega_{p,q} = \left(\frac{-p}{q}\right) \exp\left[-\left\{\tfrac{1}{4}(q-1)+\tfrac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

where (a/b) is the symbol of Legendre and Jacobi, and p' is any positive integer such that $1+pp'$ is divisible by q; that*

(1 . 73) $$A_q(n) = \sum_{(p)} \omega_{p,q}\, e^{-2np\pi i/q};$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.

Then

(1 . 74) $$p(n) = \sum_1^\nu A_q \phi_q + O(n^{-\frac{1}{4}}),$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

(1 . 75) $$\sum_1^\nu A_q \phi_q.$$

It should be observed that all the numbers A_q are real. A table of A_q from $q = 1$ to $q = 18$ is given at the end of the paper (Table II).

The proof of this theorem is given in section 5; section 4 being devoted to a number of preliminary lemmas. The proof is naturally somewhat intricate; and we trust that we have arranged it in such a form as to be readily intelligible. In section 6 we draw attention to one or two questions which our theorem, in spite of its apparent completeness, still leaves open. In section 7 we indicate some other problems in combinatory analysis and the analytic theory of numbers to which our method may be applied; and we conclude by giving some functional and numerical tables: for the latter we are indebted to Major MacMahon and Mr. H. B. C. Darling. To Major MacMahon in particular we owe many thanks for the

* See Tannery and Molk, *l.c.*, pp. 104–106, for a complete set of rules for the calculation of the value of (a/b), which is, of course, always 1 or -1. When *both* p and q are odd it is indifferent which formula is adopted.

amount of trouble he has taken over very tedious calculations. It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated.

2.

ELEMENTARY PROOF THAT $e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}$ FOR SUFFICIENTLY LARGE VALUES OF n.

2.1. In this section we give the elementary proof of the inequalities (1.32). We prove, in fact, rather more, viz., that positive constants H and K exist such that

$$\frac{H}{n} e^{2\sqrt{n}} < p(n) < \frac{K}{n} e^{2\sqrt{(2n)}} \tag{2.11}$$

for $n \geqslant 1$.* We shall use in our proof only Euler's formula (1.31) and a debased form of Stirling's theorem, easily demonstrable by quite elementary methods: the proposition that

$$n!\, e^n / n^{n+\frac{1}{2}}$$

lies between two positive constants for all positive integral values of n.

2.2. The proof of the first of the two inequalities is slightly the simpler. It is obvious that if

$$\Sigma p_r(n)\, x^n = \frac{1}{(1-x)(1-x^2)\ldots(1-x^r)}$$

so that $p_r(n)$ is the number of partitions of n into parts not exceeding r, then

$$p_r(n) = p_{r-1}(n) + p_{r-1}(n-r) + p_{r-1}(n-2r) + \ldots . \tag{2.21}$$

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* Somewhat inferior inequalities, of the type

$$2^{A[\sqrt{n}]} < p(n) < n^{B[\sqrt{n}]},$$

may be proved by *entirely* elementary reasoning; by reasoning, that is to say, which depends only on the arithmetical definition of $p(n)$ and on elementary finite algebra, and does not presuppose the notion of a limit or the definitions of the logarithmic or exponential functions.

We shall use this equation to prove, by induction, that

$$p_r(n) \geqslant \frac{rn^{r-1}}{(r!)^2}. \tag{2.22}$$

It is obvious that (2.22) is true for $r = 1$. Assuming it to be true for $r = s$, and using (2.21), we obtain

$$\begin{aligned} p_{s+1}(n) &\geqslant \frac{s}{(s!)^2}\{n^{s-1}+(n-s-1)^{s-1}+(n-2s-2)^{s-1}+\dots\} \\ &\geqslant \frac{s}{(s!)^2}\left\{\frac{n^s-(n-s-1)^s}{s(s+1)}+\frac{(n-s-1)^s-(n-2s-2)^s}{s(s+1)}+\dots\right\} \\ &= \frac{n^s}{(s+1)(s!)^2} = \frac{(s+1)n^s}{\{(s+1)!\}^2} \end{aligned}$$

This proves (2.22). Now $p(n)$ is obviously not less than $p_r(n)$, whatever the value of r. Take $r = [\sqrt{n}]$: then

$$p(n) \geqslant p_{[\sqrt{n}]}(n) \geqslant \frac{[\sqrt{n}]}{n}\,\frac{n^{[\sqrt{n}]}}{\{[\sqrt{n}]!\}^2} > \frac{H}{n}e^{2\sqrt{n}},$$

by a simple application of the degenerate form of Stirling's theorem mentioned above.

2.3. The proof of the second inequality depends upon Euler's identity. If we write

$$\Sigma q_r(n)\,x^n = \frac{1}{(1-x)^2(1-x^2)^2\dots(1-x^r)^2},$$

we have

$$q_r(n) = q_{r-1}(n)+2q_{r-1}(n-r)+3q_{r-1}(n-2r)+\dots, \tag{2.31}$$

and

$$p(n) = q_1(n-1)+q_2(n-4)+q_3(n-9)+\dots. \tag{2.32}$$

We shall first prove by induction that

$$q_r(n) \leqslant \frac{(n+r^2)^{2r-1}}{(2r-1)!\,(r!)^2}. \tag{2.33}$$

This is obviously true for $r = 1$. Assuming it to be true for $r = s$, and using (2.31), we obtain

$$\begin{aligned} q_{s+1}(n) \leqslant \frac{1}{(2s-1)!\,(s!)^2}\{&(n+s^2)^{2s-1}+2(n+s^2-s-1)^{2s-1} \\ &+3(n+s^2-2s-2)^{2s-1}+\dots\}. \end{aligned}$$

Now $$m(m-1)\,a^{m-2}b^2 \leqslant (a+b)^m - 2a^m + (a-b)^m,$$

if m is a positive integer, and a, b, and $a-b$ are positive, while if $a-b \leqslant 0$, and m is odd, the term $(a-b)^m$ may be omitted. In this inequality write

$$m = 2s+1, \quad a = n+s^2-ks-k \ (k = 0, 1, 2, \ldots), \quad b = s+1,$$

and sum with respect to k. We find that

$$(2s+1)\,2s\,(s+1)^2\,\{(n+s^2)^{2s-1} + 2(n+s^2-s-1)^{2s-1} + \ldots\} \leqslant (n+s^2+s+1)^{2s+1};$$

and so

$$q_{s+1}(n) \leqslant \frac{(n+s^2+s+1)^{2s+1}}{(2s+1)\,2s\,(s+1)^2\,(2s-1)!\,(s!)^2} \leqslant \frac{\{n+(s+1)^2\}^{2s+1}}{(2s+1)!\,\{(s+1)!\}^2}.$$

Hence (2 . 33) is true generally.

It follows from (2 . 32) that

$$p(n) = q_1(n-1) + q_2(n-4) + \ldots \leqslant \sum_1^\infty \frac{n^{2r-1}}{(2r-1)!\,(r!)^2}.$$

But, using the degenerate form of Stirling's theorem once more, we find without difficulty that

$$\frac{1}{(2r-1)!\,(r!)^2} < \frac{2^{6r}K}{4r!},$$

where K is a constant. Hence

$$p(n) < 8K \sum_1^\infty \frac{(8n)^{2r-1}}{4r!} < 8K \sum_1^\infty \frac{(8n)^{\frac{1}{2}r-1}}{r!} < \frac{K}{n}\,e^{2\sqrt{(2n)}}.$$

This is the second of the inequalities (2 . 11).

3.

APPLICATION OF A TAUBERIAN THEOREM TO THE DETERMINATION OF THE CONSTANT C.

3 . 1. The value of the constant

$$C = \lim \frac{\log p(n)}{\sqrt{n}},$$

is most naturally determined by the use of the following theorem.

If $g(x) = \Sigma a_n x^n$ *is a power-series with positive coefficients, and*

$$\log g(x) \sim \frac{A}{1-x}$$

when $x \to 1$, *then*

$$\log s_n = \log (a_0 + a_1 + \ldots + a_n) \sim 2\sqrt{(An)}$$

when $n \to \infty$.

This theorem is a special case* of Theorem C in our paper already referred to.

Now suppose that

$$g(x) = (1-x) f(x) = \Sigma \{ p(n) - p(n-1) \} x^n = \frac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots}.$$

Then $$a_n = p(n) - p(n-1)$$

is plainly positive. And

$$(3.11) \quad \log g(x) = \sum_2^\infty \log \frac{1}{1-x^\mu} = \sum_1^\infty \frac{1}{\nu} \frac{x^{2\nu}}{1-x^\nu} \sim \frac{1}{1-x} \sum_1^\infty \frac{1}{\nu^2} = \frac{\pi^2}{6(1-x)},$$

when $x \to 1$.† Hence

$$(3.12) \qquad \log p(n) = a_0 + a_1 + \ldots + a_n \sim C\sqrt{n},$$

where $C = 2\pi/\sqrt{6} = \pi\sqrt{(\frac{2}{3})}$, as in (1.53).

3.2. There is no doubt that it is possible, by "Tauberian" arguments, to prove a good deal more about $p(n)$ than is asserted by (3.12).

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* *L.c.* p. 129 (with $\alpha = 1$).

† This is a special case of much more general theorems : see

K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze", *Inaugural-Dissertation*, Berlin, 1907, pp. 25 *et seq.* ;

K. Knopp, "Über Lambertsche Reihen", *Journal für Math.*, Vol. 142, 1913, pp. 283–315 ;

G. H. Hardy, "Theorems connected with Abel's Theorem on the continuity of power series", *Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 247–265 (pp. 252, 253) ;

G. H. Hardy, "Some theorems concerning infinite series", *Math. Ann.*, Vol. 64, 1907, pp. 77–94 ;

G. H. Hardy, "Note on Lambert's series", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1913, pp. 192–198.

A direct proof is very easy : for

$$\nu x^{\nu-1}(1-x) < 1 - x^\nu < \nu(1-x),$$

$$\frac{1}{1-x} \Sigma \frac{x^{2\nu}}{\nu^2} < \log g(x) < \frac{1}{1-x} \Sigma \frac{x^{\nu+1}}{\nu^2}.$$

The functional equation satisfied by $f(x)$ shows, for example, that

$$g(x) \sim \sqrt{\left(\frac{1-x}{2\pi}\right)} \exp\left\{\frac{\pi^2}{6(1-x)}\right\},$$

a relation far more precise than (3 . 11). From this relation, and the fact that the coefficients in $g(x)$ are positive, it is certainly possible to deduce more than (3 . 12). But it hardly seems likely that arguments of this character will lead us to a proof of (1 . 41). It would be exceedingly interesting to know exactly how far they will carry us, since the method is comparatively elementary, and has a much wider range of application than the more powerful methods employed later in this paper. We must, however, reserve the discussion of this question for some future occasion.

4.

LEMMAS PRELIMINARY TO THE PROOF OF THE MAIN THEOREM.

4 . 1. We proceed now to the proof of our main theorem. The proof is somewhat intricate, and depends on a number of subsidiary theorems which we shall state as lemmas.

Lemmas concerning Farey's series.

4 . 21. The *Farey's series of order m* is the aggregate of irreducible rational fractions

$$p/q \quad (0 \leqslant p \leqslant q \leqslant m),$$

arranged in ascending order of magnitude. Thus

$$\tfrac{0}{1},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2},\ \tfrac{4}{7},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{5}{7},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{5}{6},\ \tfrac{6}{7},\ \tfrac{1}{1}$$

is the Farey's series of order 7.

LEMMA 4 . 21.—*If p/q, p'/q' are two successive terms of a Farey's series, then*

$$p'q - pq' = 1. \tag{4 . 211}$$

This is, of course, a well known theorem, first observed by Farey and

first proved by Cauchy.* The following exceedingly simple proof is due to Hurwitz.†

The result is plainly true when $m = 1$. Let us suppose it true for $m = k$; and let p/q, p'/q' be two consecutive terms in the series of order k.

Suppose now that p''/q'' is a term of the series of order $k+1$ which falls between p/q and p'/q'. Let

$$p''q-pq'' = \lambda > 0, \quad p'q''-p''q' = \mu > 0.$$

Solving these equations for p'', q'', and observing that $p'q-pq' = 1$, we obtain

$$p'' = \mu p+\lambda p', \quad q'' = \mu q+\lambda q'.$$

Consider now the aggregate of fractions

$$(\mu p+\lambda p')/(\mu q+\lambda q'),$$

where λ and μ are positive integers without common factor. All of these fractions lie between p/q and p'/q'; and all are in their lowest terms, since a factor common to numerator and denominator would divide

$$\lambda = q(\mu p+\lambda p')-p(\mu q+\lambda q'),$$

and

$$\mu = p'(\mu q+\lambda q')-q'(\mu p+\lambda p').$$

Each of them first makes its appearance in the Farey's series of order $\mu q+\lambda q'$, and the *first* of them to make its appearance must be that for which $\lambda = 1$, $\mu = 1$. Hence

$$p'' = p+p', \quad q'' = q+q',$$

$$p''q-pq'' = p'q''-p''q' = 1.$$

The lemma is consequently proved by induction.

LEMMA 4.22.—*Suppose that p/q is a term of the Farey's series of order m, and p''/q'', p'/q' the adjacent terms on the left and the right:*

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* J. Farey, "On a curious property of vulgar fractions", *Phil. Mag.*, Ser. 1, Vol. 47, 1816, pp. 385–386; A. L. Cauchy, "Démonstration d'un théorème curieux sur les nombres", *Exercices de mathématiques*, Vol. 1, 1826, pp. 114–116. Cauchy's proof was first published in the *Bulletin de la Société Philomatique* in 1816.

† A. Hurwitz, "Ueber die angenäherte Darstellung der Zahlen durch rationale Brüche", *Math. Ann.*, Vol. 44, 1894, pp. 417–436.

and let $j_{p,q}$ *denote the interval*

$$\frac{p}{q}-\frac{1}{q(q+q'')}, \quad \frac{p}{q}+\frac{1}{q(q+q')}.*$$

Then (i) *the intervals* $j_{p,q}$ *exactly fill up the continuum* (0, 1), *and* (ii) *the length of each of the parts into which* $j_{p,q}$ *is divided by* p/q† *is greater than* $1/2mq$ *and less than* $1/mq$.

(i) Since

$$\frac{1}{q(q+q')}+\frac{1}{q'(q'+q)}=\frac{1}{qq'}=\frac{p'q-pq'}{qq'}=\frac{p'}{q'}-\frac{p}{q},$$

the intervals just fill up the continuum.

(ii) Since neither q nor q' exceeds m, and one at least must be less than m, we have

$$\frac{1}{q(q+q')}>\frac{1}{2mq}.$$

Also $q+q'>m$, since otherwise $(p+p')/(q+q')$ would be a term in the series between p/q and p'/q'. Hence

$$\frac{1}{q(q+q')}<\frac{1}{mq}.$$

Standard dissection of a circle.

4.23. The following mode of dissection of a circle, based upon Lemma 4.22, is of fundamental importance for our analysis.

Suppose that the circle is defined by

$$x=Re^{2\pi i\theta} \quad (0\leqslant\theta\leqslant 1).$$

Construct the Farey's series of order m, and the corresponding intervals $j_{p,q}$. When these intervals are considered as intervals of variation of θ, and the two extreme intervals, which correspond to abutting arcs on the circle, are regarded as constituting a single interval $\xi_{1,1}$, the circle is divided into a number of arcs

$$\xi_{p,q},$$

---

* When p/q is 0/1 or 1/1, only the part of this interval inside (0, 1) is to be taken; thus $j_{0,1}$ is 0, $1/(m+1)$ and $j_{1,1}$ is $1-1/(m+1)$, 1.

† See the preceding footnote.

where q ranges from 1 to m and p through the numbers not exceeding and prime to q.* We call this dissection of the circle *the dissection* Ξ_m.

Lemmas from the theory of the linear transformation of the elliptic modular functions.

4.3. LEMMA 4.31.—*Suppose that q is a positive integer; that p is a positive integer not exceeding and prime to q; that p' is a positive integer such that $1+pp'$ is divisible by q; that $\omega_{p,q}$ is defined by the formulæ* (1.721) *or* (1.722); *that*

$$x = \exp\left(-\frac{2\pi z}{q}+\frac{2p\pi i}{q}\right), \quad x' = \exp\left(-\frac{2\pi}{qz}+\frac{2p'\pi i}{q}\right),$$

where the real part of z is positive; and that

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

Then
$$f(x) = \omega_{p,q}\sqrt{z}\exp\left(\frac{\pi}{12qz}-\frac{\pi z}{12q}\right)f(x').$$

This lemma is merely a restatement in a different notation of well known formulæ in the transformation theory.

Suppose, for example, that p is odd. If we take

$$a = p, \quad b = -q, \quad c = \frac{1+pp'}{q}, \quad d = -p',$$

so that $ad-bc = 1$; and write

$$x = q^2 = e^{2\pi i\tau}, \quad x' = Q^2 = e^{2\pi i\mathrm{T}},$$

so that
$$\tau = \frac{p}{q}+\frac{iz}{q}, \quad \mathrm{T} = \frac{p'}{q}+\frac{i}{qz};$$

then we can easily verify that

$$\mathrm{T} = \frac{c+d\tau}{a+b\tau}.$$

Also, in the notation of Tannery and Molk, we have

$$f(x) = \frac{q^{\frac{1}{12}}}{h(\tau)}, \quad f(x') = \frac{Q^{\frac{1}{12}}}{h(\mathrm{T})};$$

* $p = 0$ occurring with $q = 1$ only.

and the formula for the linear transformation of $h(\tau)$ is

$$h(\mathrm{T}) = \left(\frac{b}{a}\right) \exp\left[\left\{\tfrac{1}{4}(a-1) - \tfrac{1}{12}\left[a(b-c)+bd(a^2-1)\right]\right\}\pi i\right] \sqrt{(a+b\tau)}\, h(\tau),$$

where $\sqrt{(a+b\tau)}$ has its real part positive.* A little elementary algebra will show the equivalence of this result and ours.

The other formula for $\omega_{p,q}$ may be verified similarly, but in this case we must take

$$a = -p, \quad b = q, \quad c = -\frac{1+pp'}{q}, \quad d = p'.$$

We have included in the Appendix (Table I) a short table of some values of $\omega_{p,q}$, or rather of $(\log \omega_{p,q})/\pi i$.

LEMMA 4.32.—*The function $f(x)$ satisfies the equation*

$$(4.321)\quad f(x) = \omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}} \exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\} f(x'_{p,q}),$$

where

$$(4.322)\quad x_{p,q} = xe^{-2p\pi i/q}, \quad x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})} + \frac{2p'\pi i}{q}\right\}.$$

This is an immediate corollary from Lemma 4.31, since

$$z = \frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right), \qquad e^{-\pi z/12q} = x_{p,q}^{\frac{1}{24}},$$

$$\frac{\pi}{12qz} = \frac{\pi^2}{6q^2\log(1/x_{p,q})}, \qquad x' = \exp\left(-\frac{2\pi}{qz} + \frac{2p'\pi i}{q}\right) = x'_{p,q}.$$

If we observe that

$$f(x'_{p,q}) = 1 + p(1)x'_{p,q} + \ldots,$$

we see that, if x tends to $e^{2p\pi i/q}$ along a radius vector, or indeed any regular path which does not touch the circle of convergence, the difference

$$f(x) - \omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}} \exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\}$$

tends to zero with great rapidity. It is on this fact that our analysis is based.

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* Tannery and Molk, *l.c.*, pp. 113, 267.

Lemmas concerning the auxiliary function $F_a(x)$.

4.41. The auxiliary function $F_a(x)$ is defined by the equation

$$F_a(x) = \sum_1^\infty \psi_a(n)\, x^n.$$

where
$$\psi_a(n) = \frac{d}{dn}\,\frac{\cosh a\lambda_n - 1}{\lambda_n},$$

$$\lambda_n = \sqrt{(n-\tfrac{1}{24})}, \qquad a > 0.$$

LEMMA 4.41.—*Suppose that a cut is made along the segment* $(1, \infty)$ *in the plane of* x. *Then* $F_a(x)$ *is regular at all points inside the region thus defined.*

This lemma is an immediate corollary of a general theorem proved by Lindelöf on pp. 109 *et seq.* of his *Calcul des résidus.**

The function
$$\psi_a(z) = \frac{d}{dz}\,\frac{\cosh a\sqrt{(z-\tfrac{1}{24})} - 1}{\sqrt{(z-\tfrac{1}{24})}}$$

satisfies the conditions imposed upon it by Lindelöf, if the number which he calls a is greater than $\frac{1}{24}$; and

$$F_a(x) = \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^z}{1-e^{2\pi i z}}\,\phi(z)\,dz, \tag{4.411}$$

if
$$x = re^{i\theta}, \quad 0 < \theta < 2\pi, \quad x^z = \exp\{z(\log r + i\theta)\}.$$

4.42. LEMMA 4.42.—*Suppose that* D *is the region defined by the inequalities*

$$-\pi < -\theta_0 < \theta < \theta_0 < \pi, \quad r_0 < r, \quad 0 < r_0 < 1,$$

and that $\log(1/x)$ *has its principal value, so that* $\log(1/x)$ *is one-valued, and its square root two-valued, in* D. *Further, let*

$$\chi_a(x) = \sqrt{\{\pi \log(1/x)\}}\; x^{\frac{1}{24}} \left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\} - 1\right],$$

that value of the square root being chosen which is positive when

* Lindelöf gives references to Mellin and Le Roy, who had previously established the theorem in less general forms.

$0 < x < 1$. *Then*

$$F_a(x) - \chi_a(x)$$

*is regular inside D.**

We observe first that, when θ has a fixed value between 0 and 2π, the integral on the right-hand side of (4.411) is uniformly convergent for $\frac{1}{24} \leqslant a \leqslant a_0$. Hence we may take $a = \frac{1}{24}$ in (4.411). We thus obtain

$$F_a(x) = ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{1-e^{\frac{1}{12}\pi i - 2\pi t}} \psi_a(\tfrac{1}{24}+it)\,dt + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1-e^{\frac{1}{12}\pi i + 2\pi t}} \psi_a(\tfrac{1}{24}-it)\,dt,$$

where the $\sqrt{(it)}$ and $\sqrt{(-it)}$ which occur in $\psi_a(\frac{1}{24}+it)$ and $\psi_a(\frac{1}{24}-it)$ are to be interpreted as $e^{\frac{1}{4}\pi i}\sqrt{t}$ and $e^{-\frac{1}{4}\pi i}\sqrt{t}$ respectively. We write this in the form

$$(4.421) \qquad F_a(x) = X_a(x) + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{e^{-\frac{1}{12}\pi i + 2\pi t} - 1} \psi_a(\tfrac{1}{24}+it)\,dt$$

$$+ ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1-e^{\frac{1}{12}\pi i + 2\pi t}} \psi_a(\tfrac{1}{24}-it)\,dt$$

$$= X_a(x) + \Theta_1(x) + \Theta_2(x),$$

say, where

$$X_a(x) = ix^{\frac{1}{24}} \int_0^\infty x^{it} \psi_a(\tfrac{1}{24}+it)\,dt.$$

Now, since

$$|x^{it}| = e^{-\theta t}, \qquad |x^{-it}| = e^{\theta t},$$

the functions Θ are regular throughout the angle of Lemma 4.42. And

$$X_a(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}} \int_0^\infty e^{-\lambda t} \frac{d}{dt}\left(\frac{\cosh \mu\sqrt{t} - 1}{\sqrt{t}}\right) dt,$$

where

$$\lambda = i \log \frac{1}{x}, \qquad \mu = a\sqrt{i}.$$

The form of this integral may be calculated by supposing λ and μ positive, when we obtain

$$\int_0^\infty e^{-\lambda w^2} \frac{d}{dw}\left(\frac{\cosh \mu w - 1}{w}\right) dw = 2\lambda \int_0^\infty e^{-\lambda w^2} (\cosh \mu w - 1)\,dw$$

$$= \sqrt{(\lambda\pi)}\,(e^{\mu^2/4\lambda} - 1).$$

Hence

$$(4.422) \qquad X_a(x) = \sqrt{\{\pi \log (1/x)\}}\; x^{\frac{1}{24}} \left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\} - 1\right] = \chi_a(x),$$

and the proof of the lemma is completed.

---

* Both $F_a(x)$ and $\chi_a(x)$ are two-valued in D. The value of $F_a(x)$ contemplated is naturally that represented by the power series.

Lemmas 4.41 and 4.42 show that $x = 1$ is the sole finite singularity of the principal branch of $F_a(x)$.

4.43. LEMMA 4.43.—*Suppose that* P, θ_1, *and* A *are positive constants,* θ_1 *being less than* π. *Then*

$$|F_a(x)| < K = K(\mathrm{P}, \theta_1, A),$$

for $$0 \leqslant r \leqslant \mathrm{P}, \quad \theta_1 \leqslant \theta \leqslant 2\pi - \theta_1, \quad 0 < a \leqslant A.$$

We use K generally to denote a positive number independent of x and of a. We may employ the formula (4.411). It is plain that

$$\left| \frac{x^z}{1-e^{2\pi i z}} \right| < Ke^{-\theta_1|\eta|},$$

$$|\psi_a(z)| = \left| \frac{d}{dz} \left\{ \frac{\cosh a\sqrt{(z-\frac{1}{24})} - 1}{\sqrt{(z-\frac{1}{24})}} \right\} \right| < Ke^{K\sqrt{|\eta|}},$$

where η is the imaginary part of z. Hence

$$|F_a(x)| < K \int_{-\infty}^{\infty} e^{K\sqrt{|\eta|} - \theta_1|\eta|}\, d\eta < K.$$

4.44. LEMMA 4.44.—*Let* c *be a circle whose centre is* $x = 1$, *and whose radius* δ *is less than unity. Then*

$$|F_a(x) - \chi_a(x)| < Ka^2,$$

if x *lies in* c *and* $0 < a \leqslant A$, $K = K(\delta, A)$ *being as before independent of* x *and of* a.

If we refer back to (4.421) and (4.422), we see that it is sufficient to prove that

$$|\Theta_1(x)| < Ka^2, \qquad |\Theta_2(x)| < Ka^2;$$

and we may plainly confine ourselves to the first of these inequalities. We have

$$\Theta_1(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}} \int_0^{\infty} \frac{x^{it}}{e^{-\frac{1}{2}\pi i + 2\pi t} - 1} \frac{d}{dt} \left\{ \frac{\cosh a\sqrt{(it)} - 1}{\sqrt{t}} \right\} dt.$$

Rejecting the extraneous factor, which is plainly without importance, and integrating by parts, we obtain

$$\Theta(x) = \int_0^{\infty} \Phi(t) \frac{\cosh a\sqrt{(it)} - 1}{\sqrt{t}}\, dt,$$

where $$\Phi(t) = -\frac{ix^{it}\log x}{e^{-\frac{1}{2}\pi t+2\pi t}-1} + \frac{2\pi x^{it}e^{-\frac{1}{2}\pi t+2\pi t}}{(e^{-\frac{1}{2}\pi t+2\pi t}-1)^2}.$$

Now $|\theta| < \frac{1}{2}\pi$ and $|x^{it}| < Ke^{\frac{1}{2}\pi t}$. It follows that

$$|\Phi(t)| < Ke^{-\pi t};$$

and
$$\begin{aligned}|\Theta(x)| &< K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\,|\sinh^2 \tfrac{1}{2}a\sqrt{(it)}|\,dt\\ &< K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\,\{\cosh a\sqrt{(\tfrac{1}{2}t)} - \cos a\sqrt{(\tfrac{1}{2}t)}\}\,dt\\ &< K\int_0^\infty e^{-\pi w^2}\left(\cosh\frac{aw}{\sqrt{2}} - \cos\frac{aw}{\sqrt{2}}\right)dw\\ &= K(e^{a^2/8\pi} - e^{-a^2/8\pi}) < Ka^2.\end{aligned}$$

5.

PROOF OF THE MAIN THEOREM.

5.1. We write

$$F_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}F_{C/q}(x_{p,q}), \tag{5.11}$$

where $C = \pi\sqrt{\frac{2}{3}}$, $x_{p,q} = xe^{-2p\pi i/q}$; and

$$\Phi(x) = f(x) - \sum_q\sum_p F_{p,q}(x), \tag{5.12}$$

where the summation applies to all values of p not exceeding q and prime to q, and to all values of q such that

$$1 \leqslant q \leqslant \nu = [a\sqrt{n}], \tag{5.13}$$

a being positive and independent of n. If then

$$F_{p,q}(x) = \Sigma c_{p,q,n}x^n, \tag{5.14}$$

we have

$$p(n) - \sum_q\sum_p c_{p,q,n} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\,dx, \tag{5.15}$$

where Γ is a circle whose centre is the origin and whose radius R is less

than unity. We take

$$R = 1 - \frac{\beta}{n}, \tag{5.16}$$

where β also is positive and independent of n.

Our object is to show that the integral on the right hand side of (5.15) is of the form $O(n^{-\frac{1}{4}})$; the constant implied in the O will of course be a function of α and β. It is to be understood throughout that O's are used in this sense; $O(1)$, for instance, stands for *a function of* x, n, p, q, α, *and* β (or of some only of these variables) *which is less in absolute value than a number* $K = K(\alpha, \beta)$ *independent of* x, n, p, *and* q.

We divide up the circle Γ, by means of the dissection Ξ_ν of 4.23, into arcs $\xi_{p,q}$ each associated with a point $Re^{2p\pi i/q}$; and we denote by $\eta_{p,q}$ the arc of Γ complementary to $\xi_{p,q}$. This being so, we have

$$\begin{aligned}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\,dx &= \Sigma \int_{\xi_{p,q}} \frac{f(x) - F_{p,q}(x)}{x^{n+1}}\,dx - \Sigma \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx \\ &= \Sigma J_{p,q} - \Sigma j_{p,q},\end{aligned} \tag{5.17}$$

say. We shall prove that each of these sums is of the form $O(n^{-\frac{1}{4}})$; and we shall begin with the second sum, which only involves the auxiliary functions F.

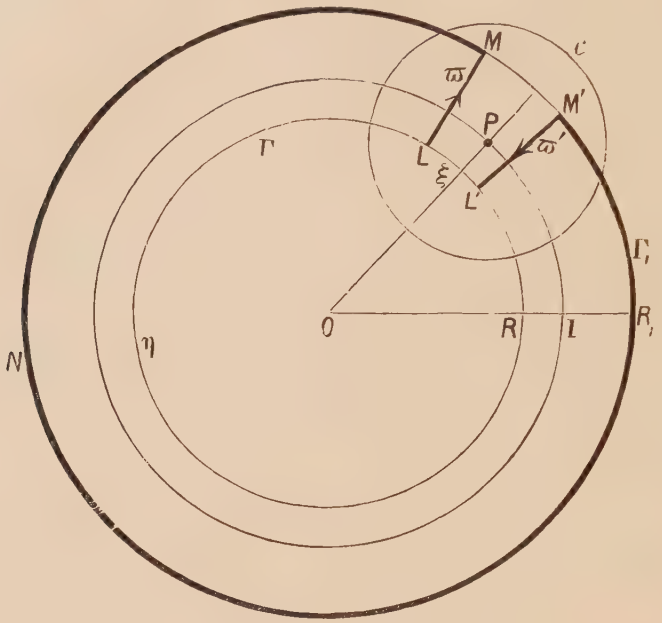
Proof that $\Sigma j_{p,q} = O(n^{-\frac{1}{4}})$.

5.21. We have, by Cauchy's theorem,

$$j_{p,q} = \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx = \int_{\zeta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx, \tag{5.211}$$

where $\zeta_{p,q}$ consists of the contour $LMNM'L'$ shown in the figure. Here L and L' are the ends of $\xi_{p,q}$, LM and $M'L'$ are radii vectores, and MNM' is part of a circle Γ_1 whose radius R_1 is greater than 1. P is the point $e^{2p\pi i/q}$; and we suppose that R_1 is small enough to ensure that all points of LM and $M'L'$ are at a distance from P less than $\frac{1}{2}$. The other circle c shown in the figure has P as its centre and radius $\frac{1}{2}$. We denote LM by $\varpi_{p,q}$, $M'L'$ by $\varpi'_{p,q}$, and MNM' by $\gamma_{p,q}$: and we write

$$j_{p,q} = \int_{\zeta_{p,q}} = \int_{\gamma_{p,q}} + \int_{\varpi_{p,q}} + \int_{\varpi'_{p,q}} = j^1_{p,q} + j^2_{p,q} + j^3_{p,q}. \tag{5.212}$$



The contribution of $\Sigma j^{1}_{p,q}$.

5 . 22. Suppose first that x lies on $\gamma_{p,q}$ and outside c. Then, in virtue of (5 . 11) and Lemma 4 . 43, we have

$$F_{p,q}(x) = O(\sqrt{q}). \tag{5 . 221}$$

If on the other hand x lies on $\gamma_{p,q}$, but inside c, we have, by (5 . 11) and Lemma 4 . 44,

$$F_{p,q}(x) - \chi_{p,q}(x) = O(q^{-\frac{3}{2}}), \tag{5 . 222}$$

where

$$\chi_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\chi_{C/q}(x_{p,q}). \tag{5 . 2221}$$

But, if we recur to the definition of $\chi_a(x)$ in Lemma 4 . 42, and observe that

$$\left|\exp\frac{a^2}{4\log(1/x)}\right| = \exp\frac{a^2\log(1/r)}{4\left[\{\log(1/r)\}^2+\theta^2\right]} < 1$$

if $x = re^{i\theta}$ and $r > 1$, we see that

$$\chi_{p,q}(x) = O(\sqrt{q}) \tag{5 . 223}$$

on the part of $\gamma_{p,q}$ in question. Hence (5.221) holds for all $\gamma_{p,q}$. It follows that

$$j^1_{p,q} = O(R_1^{-n}\sqrt{q}),$$

$$\Sigma j^1_{p,q} = O\left(R_1^{-n}\sum_q q^{\frac{3}{2}}\right) = O(n^{\frac{3}{2}}R_1^{-n}).* \tag{5.224}$$

This sum tends to zero more rapidly than any power of n, and is therefore completely trivial.

The contributions of $\Sigma j^2_{p,q}$ and $\Sigma j^3_{p,q}$.

5.231. We must now consider the sums which arise from the integrals along $\varpi_{p,q}$ and $\varpi'_{p,q}$; and it is evident that we need consider in detail only the first of these two lines. We write

$$j^2_{p,q} = \int_{\varpi_{p,q}} \frac{F_{p,q}(x)-\chi_{p,q}(x)}{x^{n+1}}\,dx + \int_{\varpi_{p,q}} \frac{\chi_{p,q}(x)}{x^{n+1}}\,dx = j'_{p,q}+j''_{p,q}, \tag{5.2311}$$

say.

In the first place we have, from (5.222),

$$j'_{p,q} = O\left(q^{-\frac{1}{2}}\int_R^{R_1}\frac{dr}{r^{n+1}}\right) = O(q^{-\frac{3}{2}}n^{-1}),$$

since

$$R^{-n} = \left(1-\frac{\beta}{n}\right)^{-n} = O(1). \tag{5.2312}$$

Thus

$$\Sigma j'_{p,q} = O\left\{n^{-1}\sum_{q<O(\sqrt{n})} q^{-\frac{1}{2}}\right\} = O(n^{-\frac{3}{4}}). \tag{5.2313}$$

5.232. In the second place we have

$$j''_{p,q} = \omega_{p,q}\,\frac{\sqrt{q}}{\pi\sqrt{2}}\int_{\varpi_{p,q}} \frac{\chi_{C/q}(x_{p,q})}{x^{n+1}}\,dx.$$

It is plain that, if we substitute y for $xe^{-2p\pi i/q}$, then write x again for y, and finally substitute for $\chi_{C/q}$ its explicit expression as an elementary

* Here, and in many passages in our subsequent argument, it is to be remembered that the number of values of p, corresponding to a given q, is less than q, and that the number of values of q is of order $\sqrt{n}$. Thus we have generally

$$\Sigma\, O(q^s) = O\left(\sum_{q<O(\sqrt{n})} q^{s+1}\right) = O(n^{\frac{1}{2}s+1}).$$

function, given in Lemma 4 . 42, we obtain

$$(5\,.\,2321)\quad \int''_{p,q} = O(\sqrt{q})\int \{E(x)-1\}\sqrt{\left(\log\frac{1}{x}\right)}\, x^{-n-\frac{23}{24}}\,dx = O(\sqrt{q})\,J,$$

say, where

$$(5\,.\,23211)\qquad E(x) = \exp\left(\frac{\pi^2}{6q^2\log(1/x)}\right),$$

and the path of integration is now a line related to $x=1$ as $\varpi_{p,q}$ is to $x = e^{2p\pi i/q}$: the line defined by $x = re^{i\theta}$, where $R \leqslant r \leqslant R_1$, and θ is fixed and (by Lemma 4 . 22) lies between $1/2q\nu$ and $1/q\nu$.

Integrating J by parts, we find

$$(5\,.\,2322)\quad \left(n-\tfrac{1}{24}\right)J = -\left[\{E(x)-1\}\sqrt{\left(\log\frac{1}{x}\right)}\, x^{-n+\frac{1}{24}}\right]_{r=R}^{r=R_1}$$
$$-\tfrac{1}{2}\int \{E(x)-1\}\left(\log\frac{1}{x}\right)^{-\frac{1}{2}} x^{-n-\frac{23}{24}}\,dx$$
$$+\frac{\pi^2}{6q^2}\int E(x)\left(\log\frac{1}{x}\right)^{-\frac{3}{2}} x^{-n-\frac{23}{24}}\,dx = J_1+J_2+J_3,$$

say.

5 . 233. In estimating J_1, J_2, and J_3, we must bear the following facts in mind.

(1) Since $|x| \geqslant R$, it follows from (5 . 2312) that $|x|^{-n} = O(1)$ throughout the range of integration.

(2) Since $1-R = \beta/n$ and $1/2q\nu < \theta < 1/q\nu$, where $\nu = [\alpha\sqrt{n}]$, we have

$$\log\left(\frac{1}{x}\right) = O\left(\frac{1}{q\sqrt{n}}\right),$$

when $r = R$, and $\qquad \dfrac{1}{\log(1/x)} = O(q\sqrt{n}),$

throughout the range of integration.

(3) Since $\qquad |E(x)| = \exp\dfrac{\pi^2\log(1/r)}{6q^2[\{\log(1/r)\}^2+\theta^2]},$

$E(x)$ is less than 1 in absolute value when $r > 1$. And, on the part of the path for which $r < 1$, it is of the form

$$\exp O\left(\frac{1}{q^2n\theta^2}\right) = \exp O(1) = O(1).$$

It is accordingly of the form $O(1)$ throughout the range of integration.

5 . 234. Thus we have, first

$$(5\,.\,2341)\quad J_1 = O(1)\,O(1)\,O(R_1^{-n}) + O(1)\,O(q^{-\frac{1}{2}}n^{-\frac{1}{4}})\,O(1) = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}),$$

secondly

$$(5\,.\,2342)\qquad J_2 = O(1)\,O(q^{\frac{1}{2}}n^{\frac{1}{4}})\int_R^{R_1}\frac{dr}{r^{n+\frac{3}{24}}} = O(q^{\frac{1}{2}}n^{-\frac{3}{4}}),$$

and thirdly

$$(5\,.\,2343)\qquad J_3 = O(q^{-2})\,O(1)\,O(q^{\frac{3}{4}}n^{\frac{3}{4}})\int_R^{R_1}\frac{dr}{r^{n+\frac{23}{24}}} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}).$$

From (5 . 2341), (5 . 2342), (5 . 2343), and (5 . 2322), we obtain

$$J = O(q^{-\frac{1}{2}}n^{-\frac{3}{4}}) + O(q^{\frac{1}{2}}n^{-\frac{7}{4}});$$

and, from (5 . 2321), $\qquad j''_{p,q} = O(n^{-\frac{3}{4}}) + O(qn^{-\frac{7}{4}})$.

Summing, we obtain

$$(5\,.\,2344)\qquad \Sigma j''_{p,q} = O\Big(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})} q\Big) + O\Big(n^{-\frac{7}{4}}\sum_{q<O(\sqrt{n})} q^2\Big)$$
$$= O(n^{-\frac{1}{4}}) + O(n^{-\frac{1}{4}}) = O(n^{-\frac{1}{4}}).$$

5 . 235. From (5 . 2311), (5 . 2313), and (5 . 2344), we obtain

$$(5\,.\,2351)\qquad \Sigma j^2_{p,q} = O(n^{-\frac{1}{4}});$$

and in exactly the same way we can prove

$$(5\,.\,2352)\qquad \Sigma j^3_{p,q} = O(n^{-\frac{1}{4}}).$$

And from (5 . 212), (5 . 224), (5 . 2351), and (5 . 2352) we obtain, finally,

$$(5\,.\,2353)\qquad \Sigma j_{p,q} = O(n^{-\frac{1}{4}}).$$

Proof that $\Sigma J_{p,q} = O(n^{-\frac{1}{4}})$.

5 . 31. We turn now to the discussion of

$$(5.311)\quad J_{p,q} = \int_{\xi_{p,q}} \frac{f(x) - F_{p,q}(x)}{x^{n+1}}\,dx$$
$$= \int_{\xi_{p,q}} \frac{f(x) - X_{p,q}(x)}{x^{n+1}}\,dx - \int_{\xi_{p,q}} \frac{F_{p,q}(x) - \chi_{p,q}(x)}{x^{n+1}}\,dx + \int_{\xi_{p,q}} \frac{\rho_{p,q}(x)}{x^{n+1}}\,dx$$
$$= J^1_{p,q} + J^2_{p,q} + J^3_{p,q},$$

say, where $$\rho_{p,q}(x) = \omega_{p,q}\sqrt{\left(\frac{q}{2\pi}\log\frac{1}{x_{p,q}}\right)}\, x_{p,q}^{\frac{1}{4}q},$$

$$X_{p,q}(x) = \chi_{p,q}(x) + \rho_{p,q}(x) = \rho_{p,q}(x)\, E(x_{p,q}),$$

$E(x)$ being defined as in (5 . 23211).

Discussion of $\Sigma J^2_{p,q}$ *and* $\Sigma J^3_{p,q}$.

5 . 32. The discussion of these two sums is, after the analysis which precedes, a simple matter. The arc $\xi_{p,q}$ is less than a constant multiple of $1/q\sqrt{n}$; and $x^{-n} = O(1)$ on $\xi_{p,q}$. Also

$$|F_{p,q}(x) - \chi_{p,q}(x)| = O(q^{-\frac{3}{2}}),$$

by (5 . 222); and

(5 . 321) $$\sqrt{\left(\log\frac{1}{x_{p,q}}\right)} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}),$$

since $$|x_{p,q}| = R = 1-(\beta/n), \qquad |am\ x_{p,q}| < 1/q\nu.$$

Hence $$J^2_{p,q} = O(q^{-\frac{5}{2}}n^{-\frac{1}{2}}),$$

(5 . 322) $$\Sigma J^2_{p,q} = O\left(n^{-\frac{1}{2}} \sum_{q<O(\sqrt{n})} q^{-\frac{3}{2}}\right) = O(n^{-\frac{1}{2}});$$

and $$J^3_{p,q} = O(q^{-1}n^{-\frac{3}{4}}),$$

(5 . 323) $$\Sigma J^3_{p,q} = O\left(n^{-\frac{3}{4}} \sum_{q<O(\sqrt{n})} 1\right) = O(n^{-\frac{1}{4}}).$$

Discussion of $\Sigma J^1_{p,q}$.

5 . 33. From (4 . 321) and (5 . 2221), we have

(5 . 331) $$f(x) - X_{p,q}(x) = \omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{4}q}\, E(x_{p,q})\, \Omega(x'_{p,q}),$$

where $$\Omega(z) = f(z) - 1 = \prod_1^\infty \left(\frac{1}{1-z^\nu}\right) - 1 = \sum_1^\infty p(\nu)\, z^\nu,$$

if $|z| < 1$, and

$$x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})} + \frac{2\pi i p'}{q}\right\}.$$

Now $$|x'_{p,q}| = \exp\left[-\frac{4\pi^2\log(1/R)}{q^2\{[\log(1/R)]^2+\theta^2\}}\right],$$

where θ is the amplitude of $x_{p,q}$. Also

$$q^2\{[\log(1/R)]^2+\theta^2\} = O\left\{q^2\left(\frac{1}{n^2}+\frac{1}{q^2n}\right)\right\} = O\left(\frac{1}{n}\right),$$

while $\log(1/R)$ is greater than a constant multiple of $1/n$. There is therefore a positive number δ, less than unity and independent of n and of q, such that

$$|x'_{p,q}| < \delta;$$

and we may write $$\Omega(x'_{p,q}) = O(|x'_{p,q}|).$$

We have therefore

$$E(x_{p,q})\,\Omega(x'_{p,q}) = O(|x'_{p,q}|^{-\frac{1}{24}})\,O(|x'_{p,q}|) = O(|x'_{p,q}|^{\frac{23}{24}}) = O(1);$$

and so, by (5 . 321),

$$f(x)-\chi_{p,q}(x) = O(\sqrt{q})\,O\left(\sqrt{\left|\log\frac{1}{x_{p,q}}\right|}\right)O(1) = O(n^{-\frac{1}{4}}).$$

And hence, as the length of $\xi_{p,q}$ is of the form $O(1/q\sqrt{n})$, we obtain

$$J^1_{p,q} = O(q^{-1}n^{-\frac{3}{4}}),$$

$$\Sigma J^1_{p,q} = O\left(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})}1\right) = O(n^{-\frac{1}{4}}). \tag{5 . 332}$$

5 . 34. From (5 . 311), (5 . 322), (5 . 323), and (5 . 332), we obtain

$$\Sigma J_{p,q} = O(n^{-\frac{1}{4}}). \tag{5 . 341}$$

Completion of the proof.

5 . 4. From (5 . 15), (5 . 17), (5 . 2353), and (5 . 341), we obtain

$$p(n)-\sum_q\sum_p c_{p,q,n} = O(n^{-\frac{1}{4}}). \tag{5 . 41}$$

But $$\sum_p c_{p,q,n} = \frac{\sqrt{q}}{\pi\sqrt{2}}\,A_q\frac{d}{dn}\,\frac{\cosh(C\lambda_n/q)-1}{\lambda_n},$$

where $$A_q = \sum_p \omega_{p,q}\,e^{-2np\pi i/q}.$$

All that remains, in order to complete the proof of the theorem, is to show that

$$\cosh(C\lambda_n/q)-1$$

may be replaced by $$\tfrac{1}{2}e^{C\lambda_n/q};$$

and in order to prove this it is only necessary to show that

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \frac{d}{dn} \frac{\frac{1}{2}e^{C\lambda_n/q} - \cosh(C\lambda_n/q) + 1}{\lambda_n} = O(n^{-\frac{1}{2}}).$$

On differentiating we find that the sum is of the form

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \left\{ O\left(\frac{1}{qn}\right) + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \right\} = O\left\{ \frac{1}{n} \sum_{q<O(\sqrt{n})} q^{\frac{1}{2}} \right\} = O(n^{-\frac{1}{4}}).$$

Thus the theorem is proved.

6.

ADDITIONAL REMARKS ON THE THEOREM.

6.1. The theorem which we have proved gives information about $p(n)$ which is in some ways extraordinarily exact. We are for this reason the more anxious to point out explicitly two respects in which the results of our analysis are incomplete.

6.21. We have proved that

$$p(n) = \Sigma A_q \phi_q + O(n^{-\frac{1}{4}}),$$

where the summation extends over the values of q specified in the theorem, for every fixed value of α; that is to say that, when α is given, a number $K = K(\alpha)$ can be found such that

$$| p(n) - \Sigma A_q \phi_q | < K n^{-\frac{1}{4}}$$

for every value of n. It follows that

$$p(n) = \{\Sigma A_q \phi_q\}, \tag{6.211}$$

where $\{x\}$ denotes the integer nearest to x, for $n \geqslant n_0$, where $n_0 = n_0(\alpha)$ is a certain function of α.

The question remains whether we can, by an appropriate choice of α, secure the truth of (6.211) for *all* values of n, and not merely for all sufficiently large values. Our opinion is that this is possible, and that it could be proved to be possible without any fundamental change in our analysis. Such a proof would however involve a very careful revision of our argument. It would be necessary to replace all formulæ involving O's by inequalities, containing only numbers expressed explicitly as functions of the various parameters employed. This process would certainly add very considerably to the length and the complexity of our argument. It

is, as it stands, sufficient to prove what is, from our point of view, of the greatest interest; and we have not thought it worth while to elaborate it further.

6.22. The second point of incompleteness of our results is of much greater interest and importance. We have not proved either that the series

$$\sum_1^\infty A_q \phi_q$$

is convergent, or that, if it is convergent, it represents $p(n)$. Nor does it seem likely that our method is one intrinsically capable of proving these results, if they are true—a point on which we are not prepared to express any definite opinion.

It should be observed in this connection that we have not even discovered anything definite concerning the order of magnitude of A_q for large values of q. We can prove nothing better than the absolutely trivial equation $A_q = O(q)$. On the other hand we cannot assert that A_q is, for an infinity of values of q, effectively of an order as great as q, or indeed even that it does not tend to zero (though of course this is most unlikely).

6.3. Our formula directs us, if we wish to obtain the exact value of $p(n)$ for a large value of n, to take a number of terms of order $\sqrt{n}$. The numerical data suggest that a considerably smaller number of terms will be equally effective; and it is easy to see that this conjecture is correct.

Let us write

$$\beta = 4\pi\sqrt{(\tfrac{2}{3})} = 4C, \quad \mu = \left[\frac{\beta\sqrt{n}}{\log n}\right],$$

and let us suppose that $\alpha < 2$. Then

$$\sum_{\mu+1}^{\nu} A_q \phi_q = \sum_{\mu+1}^{\nu} O(q^{\frac{1}{2}})\, O\left(\frac{1}{qn}\right) O(e^{C\sqrt{n}/q}) = O\left(\frac{1}{n} \sum_{\mu+1}^{\nu} \sqrt{q}\, e^{C\sqrt{n}/q}\right)$$

$$= O\left(\frac{1}{n}\int_\mu^\nu \sqrt{x}\, e^{C\sqrt{n}/x}\, dx\right),$$

since $\sqrt{q}\, e^{C\sqrt{n}/q}$ decreases steadily throughout the range of summation.*

Writing $\sqrt{n}/y$ for x, we obtain

$$O\left(n^{-\frac{1}{4}}\int_{1/\alpha}^{\sqrt{n}/\mu} y^{-\frac{5}{2}} e^{Cy}\, dy\right) = O\left\{n^{-\frac{1}{4}}\left(\frac{\sqrt{n}}{\mu}\right)^{-\frac{5}{2}} e^{C\sqrt{n}/\mu}\right\} = O\left\{n^{-\frac{1}{4}}(\log n)^{-\frac{5}{2}} e^{\frac{1}{4}\log n}\right\}$$

$$= O(\log n)^{-\frac{5}{2}} = o(1).$$

* The minimum occurs when q is about equal to $2C\sqrt{n}$.

It follows that it is enough, when n is sufficiently large, to take

$$\left[\frac{\beta\sqrt{n}}{\log n}\right]$$

terms of the series. It is probably also *necessary* to take a number of terms of order $\sqrt{n}/(\log n)$; but it is not possible to prove this rigorously without a more exact knowledge of the properties of A_q than we possess.

6.4. We add a word on certain simple approximate formulæ for $\log p(n)$ found empirically by Major MacMahon and by ourselves. Major MacMahon found that if

$$\log_{10} p(n) = \sqrt{(n+4)} - a_n, \tag{6.41}$$

then a_n is approximately equal to 2 within the limits of his table of values of $p(n)$ (Table IV). This suggested to us that we should endeavour to find more accurate formulæ of the same type. The most striking that we have found is

$$\log_{10} p(n) = \tfrac{10}{9}\{\sqrt{(n+10)} - a_n\}\ ; \tag{6.42}$$

the mode of variation of a_n is shown in Table III.

In this connection it is interesting to observe that the functiou

$$13^{-\sqrt{n}}\, p(n)$$

(which ultimately tends to infinity with exponential rapidity) is equal to ·973 for $n = 30000000000$.

7.

FURTHER APPLICATIONS OF THE METHOD.

7.1. We shall conclude with a few remarks concerning actual or possible applications of our method to other problems in Combinatory Analysis or the Analytic Theory of Numbers.

The class of problems in which the method gives the most striking results may be defined as follows. Suppose that $q(n)$ is the coefficient of x^n in the expansion of $F(x)$, where $F(x)$ is a function of the form

$$\frac{\{f(\pm x^a)\}^\alpha \{f(\pm x^{a'})\}^{\alpha'} \dots}{\{f(\pm x^b)\}^\beta \{f(\pm x^{b'})\}^{\beta'} \dots}\ ;* \tag{7.11}$$

* Since

$$f(-x) = \frac{\{f(x^2)\}^3}{f(x)\, f(x^4)},$$

the arguments with a negative sign may be eliminated if this is desired.

$f(x)$ being the function considered in this paper, the a's, b's, α's, and β's being positive integers, and the number of factors in numerator and denominator being finite; and suppose that $|F(x)|$ tends exponentially to infinity when x tends in an appropriate manner to some or all of the points $e^{2p\pi i/q}$. Then our method may be applied in its full power to the asymptotic study of $q(n)$, and yields results very similar to those which we have found concerning $p(n)$.

Thus, if

$$F(x) = \frac{f(x)}{f(x^2)} = (1+x)(1+x^2)(1+x^3)\ldots = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots},$$

so that $q(n)$ is the number of partitions of n into odd parts, or into unequal parts*, we find that

$$q(n) = \frac{1}{\sqrt{2}}\frac{d}{dn}J_0\left[i\pi\sqrt{\left\{\tfrac{1}{3}(n+\tfrac{1}{24})\right\}}\right] + \sqrt{2}\cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{9}\pi\right)\frac{d}{dn}J_0\left[\tfrac{1}{3}i\pi\sqrt{\left\{\tfrac{1}{3}(n+\tfrac{1}{24})\right\}}\right] + \ldots.$$

The error after $[\alpha\sqrt{n}]$ terms is of the form $O(1)$. We are not in a position to assert that the *exact* value of $q(n)$ can always be obtained from the formula (though this is probable); but the error is certainly bounded.

If $$F(x) = \frac{f(x^2)}{f(-x)} = \frac{f(x)f(x^4)}{\{f(x^2)\}^2} = (1+x)(1+x^3)(1+x^5)\ldots,$$

so that $q(n)$ is the number of partitions of n into parts which are both odd and unequal, then

$$q(n) = \frac{d}{dn}J_0\left[i\pi\sqrt{\left\{\tfrac{1}{6}(n-\tfrac{1}{24})\right\}}\right] + 2\cos\left(\tfrac{2}{3}n\pi - \tfrac{2}{9}\pi\right)\frac{d}{dn}J_0\left[\tfrac{1}{3}i\pi\sqrt{\left\{\tfrac{1}{6}(n-\tfrac{1}{24})\right\}}\right] + \ldots.$$

The error is again bounded (and probably tends to zero).

If $$F(x) = \frac{\{f(x)\}^2}{f(x^2)} = \frac{1}{1-2x+2x^4-2x^9+\ldots},$$

$q(n)$ has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of a simple ϑ-function, of particular

* Cf. MacMahon, *loc. cit.*, p. 11. We give at the end of the paper a table (Table V) of the values of $q(n)$ up to $n = 100$. This table was calculated by Mr. Darling.

interest. In this case we find

$$q(n) = \frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\sqrt{n}} + \frac{\sqrt{3}}{2\pi}\cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{6}\pi\right)\frac{d}{dn}\frac{e^{\frac{1}{3}\pi\sqrt{n}}}{\sqrt{n}} + \dots .$$

The error here is (as in the partition problem) of order $O(n^{-\frac{1}{4}})$, and the exact value can always be found from the formula.

7.2. The method may also be applied to products of the form (7.11) which have (to put the matter roughly) no exponential infinities. In such cases the approximation is of a much less exact character. On the other hand the problems of this character are of even greater arithmetical interest.

The standard problem of this category is that of the representation of a number as the sum of s squares, s being any positive integer odd or even.* We must reserve the application of our method to this problem for another occasion; but we can indicate the character of our main result as follows.

If $r_s(n)$ is the number of representations of n as the sum of s squares, we have

$$F(x) = \Sigma r_s(n)\, x^n = (1+2x+2x^4+\dots)^s = \frac{\{f(x^2)\}^s}{\{f(-x)\}^{2s}} = \frac{\{f(x)\}^{2s}\{f(x^4)\}^{2s}}{\{f(x^2)\}^{5s}}.$$

We find that

$$r_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\, n^{\frac{1}{2}s-1}\, \Sigma\frac{c_q}{q^{\frac{1}{2}s}} + O(n^{\frac{1}{4}s}), \tag{7.21}$$

where c_q is a function of q and of n of the same general type as the function A_q of this paper. The series

$$\Sigma\frac{c_q}{q^{\frac{1}{2}s}} \tag{7.22}$$

is absolutely convergent for sufficiently large values of s, and the summation in (7.21) may be regarded indifferently as extended over all values of q or only over a range $1 \leqslant q \leqslant \alpha\sqrt{n}$. It should be observed that the series (7.22) is quite different in form from any of the infinite series which are already known to occur in connection with this problem.

7.3. There is also a wide range of problems to which our methods

* As is well known, the arithmetical difficulties of the problem are much greater when s is odd.

are *partly* applicable. Suppose, for example, that

$$F(x) = \Sigma\, p^2(n)\, x^n = \frac{1}{(1-x)(1-x^4)(1-x^9)\dots},$$

so that $p^2(n)$ is the number of partitions of n into *squares*. Then $F(x)$ is not an elliptic modular function; it possesses no general transformation theory: and the full force of our method cannot be applied. We can still, however, apply some of our preliminary methods. Thus the "Tauberian" argument shows that

$$\log p^2(n) \sim 2^{-\frac{1}{3}} 3\pi^{\frac{1}{3}} \left\{\zeta\left(\tfrac{3}{2}\right)\right\}^{\frac{2}{3}} n^{\frac{1}{3}}.$$

And although there is no general transformation theory, there is a formula which enables us to specify the nature of the singularity at $x = 1$. This formula is

$$\frac{1}{f(e^{-\pi z})} = 2\sqrt{\left(\frac{\pi}{z}\right)} \exp\left\{\frac{2\pi}{\sqrt{z}}\, \zeta\left(-\tfrac{1}{2}\right)\right\} \times \prod_1^\infty \left\{1 - 2e^{-2\pi\sqrt{(n/z)}} \cos 2\pi\sqrt{(n/z)} + e^{-4\pi\sqrt{(n/z)}}\right\}.$$

By the use of this formula, in conjunction with Cauchy's theorem, it is certainly possible to obtain much more precise information about $p^2(n)$, and in particular the formula

$$p^2(n) \sim 3^{-\frac{1}{2}} (4\pi n)^{-\frac{7}{6}} \left\{\zeta\left(\tfrac{3}{2}\right)\right\}^{\frac{2}{3}} e^{2^{-\frac{1}{3}} 3\pi^{\frac{1}{3}} \{\zeta(\frac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}}.$$

The corresponding formula for $p^s(n)$, the number of partitions of n into perfect s-th powers, is

$$p^s(n) \sim (2\pi)^{-\frac{1}{2}(s+1)} \sqrt{\left(\frac{s}{s+1}\right)}\, k n^{\frac{1}{s+1} - \frac{3}{2}}\, e^{(s+1)\, k n^{1/(s+1)}},$$

where

$$k = \left\{\frac{1}{s}\, \Gamma\left(1+\frac{1}{s}\right) \zeta\left(1+\frac{1}{s}\right)\right\}^{\frac{s}{s+1}}.$$

[The series in (7 . 21) may be written in the form

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\, n^{\frac{1}{2}s-1} \sum_{p,\,q} \frac{\omega^s_{p,\,q}}{q^{\frac{1}{2}s}}\, e^{-np\pi i/q},$$

where $\omega_{p,\,q}$ is always one of the five numbers 0, $e^{\frac{1}{4}\pi i}$, $e^{-\frac{1}{4}\pi i}$, $-e^{\frac{1}{4}\pi i}$, $-e^{-\frac{1}{4}\pi i}$. When s is even it begins

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\, n^{\frac{1}{2}s-1} \left\{1^{-\frac{1}{2}s} + 2\cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{4}s\pi\right) 2^{-\frac{1}{2}s} + 2\cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{2}s\pi\right) 3^{-\frac{1}{2}s} + \dots\right\},$$

It has been proved by Ramanujan that the series gives an *exact* representation of $r_s(n)$ when $s = 4, 6, 8$; and by Hardy that this is also true when $s = 3, 5, 7$. See Ramanujan, "On certain trigonometrical sums and their applications in the Theory of Numbers"; Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five or seven".*—*Added April 19th*, 1918.]

TABLE I: $\omega_{p,q}$.

p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$
1	1	0	3	11	3/22	8	15	7/18
1	2	0	4	,,	3/22	11	,.	−19/90
1	3	1/18	5	,,	−5/22	13	,,	−7/18
2	,,	−1/18	6	,,	5/22	14	,,	−1/90
1	4	1/8	7	,,	−3/22	1	16	−29/32
3	,,	−1/8	8	,,	−3/22	3	,,	−27/32
1	5	1/5	9	,,	−5/22	5	,,	−5/32
2	,,	0	10	,,	−15/22	7	,,	−3/32
3	,,	0	1	12	55/72	9	,,	3/32
4	,,	−1/5	5	,,	−1/72	11	,,	5/32
1	6	5/18	7	,,	1/72	13	,,	27/32
5	,,	−5/18	11	,,	−55/72	15	,,	29/32
1	7	5/14	1	13	11/13	1	17	−14/17
2	,,	1/14	2	,,	4/13	2	,,	8/17
3	,,	−1/14	3	,,	1/13	3	,,	5/17
4	,,	1/14	4	,,	−1/13	4	,,	0
5	,,	−1/14	5	,,	0	5	,,	1/17
6	,,	−5/14	6	,,	−4/13	6	,,	5/17
1	8	7/16	7	,,	4/13	7	,,	1/17
3	,,	1/16	8	,,	0	8	,,	−8/17
5	,,	−1/16	9	,,	1/13	9	,,	8/17
7	,.	−7/16	10	,,	−1/13	10	,,	−1/17
1	9	14/27	11	,,	−4/13	11	,,	−5/17
2	,,	4/27	12	,,	−11/13	12	,,	−1/17
4	,,	−4/27	1	14	13/14	13	,,	0
5	,,	4/27	3	,,	3/14	14	,,	−5/17
7	,,	−4/27	5	,,	3/14	15	,,	−8/17
8	,,	−14/27	9	,,	−3/14	16	,,	14/17
1	10	3/5	11	,,	−3/14	1	18	−20/27
3	,,	0	13	,,	−13/14	5	,,	2/27
7	,,	0	1	15	1/90	7	,,	−2/27
9	,,	−3/5	2	,,	7/18	11	,,	2/27
1	11	15/22	4	,,	19/90	13	,,	−2/27
2	,,	5/22	7	,,	−7/18	17	,,	20/27

* The first paper is in course of publication in the *Transactions of the Cambridge Philosophical Society*. An account of the second is to appear shortly in the *Proceedings of the National Academy of Sciences* (Washington, D.C.); see also *Records of Proceedings at Meetings*, March 1918.

TABLE II: A_q.

$A_1 = 1.$

$A_2 = \cos n\pi.$

$A_3 = 2\cos(\frac{2}{3}n\pi - \frac{1}{18}\pi).$

$A_4 = 2\cos(\frac{1}{2}n\pi - \frac{1}{8}\pi).$

$A_5 = 2\cos(\frac{2}{5}n\pi - \frac{1}{5}\pi) + 2\cos\frac{4}{5}n\pi.$

$A_6 = 2\cos(\frac{1}{3}n\pi - \frac{5}{18}\pi).$

$A_7 = 2\cos(\frac{2}{7}n\pi - \frac{5}{14}\pi) + 2\cos(\frac{4}{7}n\pi - \frac{1}{14}\pi) + 2\cos(\frac{6}{7}n\pi + \frac{1}{14}\pi).$

$A_8 = 2\cos(\frac{1}{4}n\pi - \frac{7}{16}\pi) + 2\cos(\frac{3}{4}n\pi - \frac{1}{16}\pi).$

$A_9 = 2\cos(\frac{2}{9}n\pi - \frac{1}{2}\frac{4}{7}\pi) + 2\cos(\frac{4}{9}n\pi - \frac{4}{2}\frac{1}{7}\pi) + 2\cos(\frac{8}{9}n\pi + \frac{4}{2}\frac{1}{7}\pi).$

$A_{10} = 2\cos(\frac{1}{5}n\pi - \frac{3}{8}\pi) + 2\cos\frac{3}{5}n\pi.$

$A_{11} = 2\cos(\frac{2}{11}n\pi - \frac{1}{2}\frac{5}{3}\pi) + 2\cos(\frac{4}{11}n\pi - \frac{5}{2}\frac{1}{2}\pi) + 2\cos(\frac{6}{11}n\pi - \frac{3}{2}\frac{1}{2}\pi) + 2\cos(\frac{8}{11}n\pi - \frac{3}{2}\frac{2}{2}\pi) + 2\cos(\frac{1}{1}\frac{0}{1}n\pi + \frac{5}{2}\frac{1}{2}\pi).$

$A_{12} = 2\cos(\frac{1}{6}n\pi - \frac{5}{7}\frac{5}{2}\pi) + 2\cos(\frac{5}{6}n\pi + \frac{1}{7}\frac{1}{2}\pi).$

$A_{13} = 2\cos(\frac{2}{13}n\pi - \frac{1}{1}\frac{1}{3}\pi) + 2\cos(\frac{4}{13}n\pi - \frac{4}{13}\pi) + 2\cos(\frac{6}{13}n\pi - \frac{1}{13}\pi) + 2\cos(\frac{8}{13}n\pi + \frac{1}{13}\pi) + 2\cos\frac{1}{13}\frac{0}{0}n + 2\cos(\frac{1}{1}\frac{2}{3}n\pi + \frac{4}{13}\pi).$

$A_{14} = 2\cos(\frac{1}{7}n\pi - \frac{1}{1}\frac{3}{4}\pi) + 2\cos(\frac{3}{7}n\pi - \frac{3}{14}\pi) + 2\cos(\frac{5}{7}n\pi - \frac{3}{14}\pi).$

$A_{15} = 2\cos(\frac{2}{15}n\pi - \frac{1}{9}\frac{1}{0}\pi) + 2\cos(\frac{4}{15}n\pi - \frac{7}{18}\pi) + 2\cos(\frac{8}{15}n\pi - \frac{1}{9}\frac{9}{0}\pi) + 2\cos(\frac{1}{1}\frac{4}{5}n\pi + \frac{7}{18}\pi).$

$A_{16} = 2\cos(\frac{1}{8}n\pi + \frac{2}{3}\frac{9}{2}\pi) + 2\cos(\frac{3}{8}n\pi + \frac{2}{3}\frac{7}{2}\pi) + 2\cos(\frac{5}{8}n\pi + \frac{5}{3}\frac{1}{2}\pi) + 2\cos(\frac{7}{8}n\pi + \frac{3}{3}\frac{1}{2}\pi).$

$A_{17} = 2\cos(\frac{2}{17}n\pi + \frac{1}{1}\frac{4}{7}\pi) + 2\cos(\frac{4}{17}n\pi - \frac{8}{17}\pi) + 2\cos(\frac{6}{17}n\pi - \frac{5}{17}\pi) + 2\cos(\frac{8}{17}n\pi + 2\cos(\frac{1}{1}\frac{0}{7}n\pi - \frac{1}{17}\pi) + 2\cos(\frac{1}{1}\frac{2}{7}n\pi - \frac{5}{17}n\pi) + 2\cos(\frac{1}{1}\frac{4}{7}n\pi - \frac{1}{17}\pi) + 2\cos(\frac{1}{1}\frac{6}{7}n\pi + \frac{8}{17}\pi).$

$A_{18} = 2\cos(\frac{1}{9}n\pi + \frac{2}{2}\frac{0}{7}\pi) + 2\cos(\frac{5}{9}n\pi - \frac{2}{2}\frac{2}{7}\pi) + 2\cos(\frac{7}{9}n\pi + \frac{2}{2}\frac{2}{7}\pi).$

It may be observed that

$A_5 = 0$ $\left(n \equiv 1, 2 \pmod 5\right)$, $\quad A_7 = 0$ $\left(n \equiv 1, 3, 4 \pmod 7\right)$,

$A_{10} = 0$ $\left(n \equiv 1, 2 \pmod 5\right)$, $\quad A_{11} = 0$ $\left(n \equiv 1, 2, 3, 5, 7 \pmod{11}\right)$,

$A_{13} = 0$ $\left(n \equiv 2, 3, 5, 7, 9, 10 \pmod{13}\right)$, $\quad A_{14} = 0$ $\left(n \equiv 1, 3, 4 \pmod 7\right)$,

$A_{16} = 0$ $\left(n \equiv 0 \pmod 2\right)$, $\quad A_{17} = 0$ $\left(n \equiv 1, 3, 4, 6, 7, 9, 13, 14 \pmod{17}\right)$;

while A_1, A_2, A_3, A_4, A_6, A_8, A_9, A_{12}, A_{15}, and A_{18} never vanish.

TABLE III: $\log_{10} p(n) = \frac{1}{9}\pi\sqrt{\{(n+10) - a_n\}}$.

n	a_n	n	a_n
1	3·317	10000	4·148
3	3·176	30000	4·364
10	3·011	100000	4·448
30	2·951	300000	4·267
100	3·036	1000000	3·554
300	3·237	3000000	2·072
1000	3·537	10000000	−1·188
3000	3·838	30000000	−6·796
		∞	$-\infty$

TABLE IV*: $p(n)$.

1 ...	1	39 ...	31185	77 ...	10619863	115 ...	1064144451
2 ...	2	40 ...	37338	78 ...	12132164	116 ...	1188908248
3 ...	3	41 ...	44583	79 ...	13848650	117 ...	1327710076
4 ...	5	42 ...	53174	80 ...	15796476	118 ...	1482074143
5 ...	7	43 ...	63261	81 ...	18004327	119 ...	1653668665
6 ...	11	44 ...	75175	82 ...	20506255	120 ...	1844349560
7 ...	15	45 ...	89134	83 ...	23338469	121 ...	2056148051
8 ...	22	46 ...	105558	84 ...	26543660	122 ...	2291320912
9 ...	30	47 ...	124754	85 ...	30167357	123 ...	2552338241
10 ...	42	48 ...	147273	86 ...	34262962	124 ...	2841940500
11 ...	56	49 ...	173525	87 ...	38887673	125 ...	3163127352
12 ...	77	50 ...	204226	88 ...	44108109	126 ...	3519222692
13 ...	101	51 ...	239943	89 ...	49995925	127 ...	3913864295
14 ...	135	52 ...	281589	90 ...	56634173	128 ...	4351078600
15 ...	176	53 ...	329931	91 ...	64112359	129 ...	4835271870
16 ...	231	54 ...	386155	92 ...	72533807	130 ...	5371315400
17 ...	297	55 ...	451276	93 ...	82010177	131 ...	5964539504
18 ...	385	56 ...	526823	94 ...	92669720	132 ...	6620830889
19 ...	490	57 ...	614154	95 ...	104651419	133 ...	7346629512
20 ...	627	58 ...	715220	96 ...	118114304	134 ...	8149040695
21 ...	792	59 ...	831820	97 ...	133230930	135 ...	9035836076
22 ...	1002	60 ...	966467	98 ...	150198136	136 ...	10015581680
23 ...	1255	61 ...	1121505	99 ...	169229875	137 ...	11097645016
24 ...	1575	62 ...	1300156	100 ...	190569292	138 ...	12292341831
25 ...	1958	63 ...	1505499	101 ...	214481126	139 ...	13610949895
26 ...	2436	64 ...	1741630	102 ...	241265379	140 ...	15065878135
27 ...	3010	65 ...	2012558	103 ...	271248950	141 ...	16670689208
28 ...	3718	66 ...	2323520	104 ...	304801365	142 ...	18440293320
29 ...	4565	67 ...	2679689	105 ...	342325709	143 ...	20390982757
30 ...	5604	68 ...	3087735	106 ...	384276336	144 ...	22540654445
31 ...	6842	69 ...	3554345	107 ...	431149389	145 ...	24908858009
32 ...	8349	70 ...	4087968	108 ...	483502844	146 ...	27517052599
33 ...	10143	71 ...	4697205	109 ...	541946240	147 ...	30388671978
34 ...	12310	72 ...	5392783	110 ...	607163746	148 ...	33549419497
35 ...	14883	73 ...	6185689	111 ...	679903203	149 ...	37027355200
36 ...	17977	74 ...	7089500	112 ...	761002156	150 ...	40853235313
37 ...	21637	75 ...	8118264	113 ...	851376628	151 ...	45060624582
38 ...	26015	76 ...	9289091	114 ...	952050665	152 ...	49686288421

* The numbers in this table were calculated by Major MacMahon, by means of the recurrence formulæ obtained by equating coefficients in the identity

$$(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots)\sum_0^\infty p(n)\,x^n = 1.$$

We have verified the table by direct calculation up to $n = 158$. Our calculation of $p(200)$ from the asymptotic formula then seemed to render further verification unnecessary.

TABLE IV.—*Continued.*

153 ... 54770336324	165 ... 172389800255	177 ... 522115831195	189 ... 1527273599625
154 ... 60356673280	166 ... 189334822579	178 ... 571701605655	190 ... 1667727404093
155 ... 66493182097	167 ... 207890420102	179 ... 625846753120	191 ... 1820701100652
156 ... 73232243759	168 ... 228204732751	180 ... 684957390936	192 ... 1987276856363
157 ... 80630964769	169 ... 250438925115	181 ... 749474411781	193 ... 2168627105469
158 ... 88751778802	170 ... 274768617130	182 ... 819876908323	194 ... 2366022741845
159 ... 97662728555	171 ... 301384802048	183 ... 896684817527	195 ... 2580840212973
160 ... 107438159466	172 ... 330495499613	184 ... 980462880430	196 ... 2814570987591
161 ... 118159068427	173 ... 362326859895	185 ... 1071823774337	197 ... 3068829878530
162 ... 129913904637	174 ... 397125074750	186 ... 1171432692373	198 ... 3345365983698
163 ... 142798995930	175 ... 435157697830	187 ... 1280011042268	199 ... 3646072432125
164 ... 156919475295	176 ... 476715857290	188 ... 1398341745571	200 .. 3972999029388

TABLE V: $q(n)$.*

n	c_n	n	c_n	n	c_n	n	c_n
1 ...	1	26 ...	165	51 ...	4097	76 ...	53250
2 ...	1	27 ...	192	52 ...	4582	77 ...	58499
3 ...	2	28 ...	222	53 ...	5120	78 ...	64234
4 ...	2	29 ...	256	54 ...	5718	79 ...	70488
5 ...	3	30 ...	296	55 ...	6378	80 ...	77312
6 ...	4	31 ...	340	56 ...	7108	81 ...	84756
7 ...	5	32 ...	390	57 ...	7917	82 ...	92864
8 ...	6	33 ...	448	58 ...	8808	83 ...	101698
9 ...	8	34 ...	512	59 ...	9792	84 ...	111322
10 ...	10	35 ...	585	60 ...	10880	85 ...	121792
11 ...	12	36 ...	668	61 ...	12076	86 ...	133184
12 ...	15	37 ...	760	62 ...	13394	87 ...	145578
13 ...	18	38 ...	864	63 ...	14848	88 ...	159046
14 ...	22	39 ...	982	64 ...	16444	89 ...	173682
15 ...	27	40 ...	1113	65 ...	18200	90 ...	189586
16 ...	32	41 ...	1260	66 ...	20132	91 ...	206848
17 ..	38	42 ...	1426	67 ...	22250	92 ...	225585
18 ...	46	43 ...	1610	68 ...	24576	93 ...	245920
19 ...	54	44 ...	1816	69 ...	27130	94 ...	267968
20 ...	64	45 ...	2048	70 ...	29927	95 ...	291874
21 ...	76	46 ...	2304	71 ...	32992	96 ...	317788
22 ...	89	47 ...	2590	72 ...	36352	97 ...	345856
23 ...	104	48 ...	2910	73 ...	40026	98 ...	376256
24 ...	122	49 ...	3264	74 ...	44046	99 ...	409174
25 ...	142	50 ...	3658	75 ...	48446	100 ...	444793

* We are indebted to Mr. Darling for this table.

THE HARMONIC FUNCTIONS ASSOCIATED WITH THE PARABOLIC CYLINDER

(SECOND PAPER.)

By G. N. WATSON.

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1. It has been shewn by Weber* that the functions associated with the parabolic cylinder in harmonic analysis are solutions of a differential equation of the form

$$y''+Zy=0,$$

where Z is a quadratic function of the independent variable z. There is no loss of real generality in taking the equation to be

$$y''+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y=0;$$

and, when n is a positive integer, a solution of the equation is

$$(-)^n\, e^{\frac{1}{4}z^2}\frac{d^n}{dz^n}(e^{-\frac{1}{2}z^2}).$$

This function is intimately connected with Hermite's† polynomial

$$U_n(z)=e^{z^2}\frac{d^n}{dz^n}(e^{-z^2}).$$

References to other investigations of the solutions of the differential equations are to be found in papers‡ published in the *Proceedings* in 1903 and 1910. It should be stated here that the standard solution of the

* *Math. Ann.*, Vol. 1 (1869), pp. 1–36.

† *Comptes rendus*, t. 58 (1864), pp. 93–97, 266–273.

‡ Whittaker, *Proceedings*, Ser. 1, Vol. 35 (1903), pp. 417–427; Watson, *Proceedings*, Ser. 2, Vol. 8 (1910), pp. 393–421. See *Encyclopédie des Sciences Math.*, Vol. 2, t. 5, p. 325. Reference may also be made to the work of Markoff, *Bulletin de l'Académie des Sciences de St. Pétersbourg*, Sér. 5, t. 9 (1898), pp. 435–446, and A. Milne, *Proc. Edinburgh Math. Soc.*, Vol. 33 (1915), pp. 53–59, concerning the zeros of $D_n(z)$.

equation, when n is unrestricted, is taken by Whittaker to be

$$D_n(z) = -\frac{\Gamma(n+1)}{2\pi i} e^{-\frac{1}{4}z^2} z^n \int_{\infty z}^{(0+)} e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1}\, dt,$$

and other solutions are $D_n(-z)$, $D_{-n-1}(\pm iz)$.

When n is a positive integer, then

$$D_n(z) = (-)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n} (e^{-\frac{1}{2}z^2}) = (-)^n 2^{-\frac{1}{2}n} e^{-\frac{1}{4}z^2} U_n(z/\sqrt{2}).$$

The most remarkable discovery which has been made concerning the function $D_n(z)$ is contained in Adamoff's* approximation, namely that, when n is a large integer and x is positive,

$$D_n(x) = 2^{\frac{1}{2}} n^{\frac{1}{2}n} e^{-\frac{1}{2}n} \left[\cos(xn^{\frac{1}{2}} - \tfrac{1}{2}n\pi) + n^{-\frac{1}{2}} w_n(x)\right],$$

where

$$w_n(x) = e^{\frac{1}{4}x^2} \vartheta_n(x)/(x\sqrt{\pi}),$$

and $|\vartheta_n(x)| < 3{\cdot}35\ldots$, when $n > 2$; and $w_n(0) = 0$ or $\frac{1}{6}\vartheta/n$ (where $-1 < \vartheta < 1$), according as n is odd or even.

In my former paper, I shewed that Adamoff's investigation would be extended to $D_n(z)$, when z is complex, provided that $|I(z)|$ is not too large; and this result made it easy to give a formal proof of a theorem enunciated by Hermite and Whittaker, concerning the expansibility of an arbitrary analytic function in the form $\Sigma a_n D_n(z)$:—a result analogous to Neumann's expansions in series of Legendre polynomials and Bessel functions.

In this paper, I investigate various types of asymptotic formulæ and expansions of $D_n(z)$, by the method of steepest descents; the results contained in the paper are therefore to be associated with those previously obtained by Debye† and by myself‡ for Bessel functions.

It should be stated here that the method of steepest descents§ is a mode of discussing the integral $\int e^{nf(s)} \phi(s)\, ds$ by choosing a contour which passes through a zero of $f'(s)$, and which

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* *Ann. de l'Inst. Polytechnique de St. Pétersbourg*, t. 5 (1906), pp. 127–143. More recently, Perron, *Archiv der Math. u. Phys.*, Ser. 3, Vol. 22 (1914), pp. 329–340, has given some approximate formulæ for the confluent hypergeometric function of which $D_n(z)$ is a special form, but the limits of error in the formulæ are not investigated with the preciseness of the earlier work of Adamoff.

† *Math. Ann.*, Vol. 67 (1909), pp. 535–558; *Münchener Sitzungsberichte* [5], 1910.

‡ *Proceedings*, Ser. 2, Vol. 16 (1917), pp. 150–174. The present investigation is much more closely connected with this paper than with the paper of which it is nominally a sequel.

§ The method is to be traced to a posthumous paper by Riemann, *Werke*, 1876, p. 405. See also Brillouin, *Ann. de l'École Norm. Sup.*, Sér. 3, t. 33 (1916), pp. 17–69 Riemann's method was a combination of the method of steepest descents and the method of stationary phase.

is such that $If(s)$ is constant on the contour; so that $Rf(s)$ falls away from its maximum as rapidly as possible, $f(s)$ being monogenic. The name is derived (like the German "Methode der Sattelpunkte" and the French "Méthode du Col") from a consideration of the surface on which the coordinates of any point are $R(s)$, $I(s)$, $Rf(s)$; the contour is the plan of a curve drawn on the surface so as to pass through a saddlepoint and at all other points on it to be as steep as possible.

2. *Whittaker's integral for* $D_n(2\xi\sqrt{n})$.

It is convenient to take a new variable in place of z, the argument of $D_n(z)$. We shall write throughout $z = 2\xi\sqrt{n}$, $x = 2\xi\sqrt{n}$, where ξ is real; it will be supposed that n is positive (except in §§ 10–14, 16, 17, in which it is assumed merely that $R(n) > 0$ and $|\arg n| < \frac{1}{2}\pi$), so that, with these exceptions, x is also real.

In Whittaker's integral, already quoted, we take a new parametric variable s, defined by the equation

$$t = -zs\sqrt{n},$$

where $|\arg s| \leqslant \pi$ on the path of integration. This gives

$$D_n(z) = \frac{\Gamma(n+1)}{2\pi i\, n^{\frac{1}{2}n}} \int_{-\infty}^{(0+)} s^{-n-1} \exp\{-\tfrac{1}{4}z^2 + n^{\frac{1}{2}}zs - \tfrac{1}{2}ns^2\}\, ds$$

$$= \pi^{-1}(e/n)^{\frac{1}{2}n}\, \Gamma(n+1)\, \Delta_n(\xi),$$

if

$$\Delta_n(\xi) = \frac{1}{2i} \int_{-\infty}^{(0+)} \exp\{-n(\xi^2 + \tfrac{1}{2} + \log s - 2\xi s + \tfrac{1}{2}s^2)\}\, s^{-1}\, ds$$

$$= \frac{1}{2i} \int_{-\infty}^{(0+)} e^{-n\tau} s^{-1}\, ds,$$

where

$$\tau \equiv \xi^2 + \tfrac{1}{2} + \log s - 2\xi s + \tfrac{1}{2}s^2.$$

We proceed to discuss the properties of $D_n(z)$ by means of the slightly simpler function $\Delta_n(\xi)$.

3. *Inequalities satisfied by* $\Delta_n(\xi)$ *when* $\xi \geqslant 1$.

We shall now apply the principles of the method of steepest descents to the contour integral for $\Delta_n(\xi)$, introduced in § 2, in the special case in which $\xi \geqslant 1$.

Write $\xi = \cosh\alpha$, where $\alpha \geqslant 0$; then the value of τ becomes

$$\cosh^2\alpha + \tfrac{1}{2} + \log s - 2s\cosh\alpha + \tfrac{1}{2}s^2.$$

Evidently τ, *qua* function of s, is stationary when

$$(1/s)-2\cosh\alpha+s=0,$$

i.e., when $$s=e^{\pm\alpha}.$$

At these points τ is real, and so we choose (if possible) a contour on which $I(\tau)=0$.

Writing $s=re^{i\theta}$, we see that the contour must be given by the whole or part of the curve whose equation in polar coordinates is

$$\theta-2r\sin\theta\cosh\alpha+r^2\sin\theta\cos\theta=0.$$

This equation is satisfied, either if $\theta=0$, or if

$$r=\sec\theta\{\cosh\alpha\pm\sqrt{(\cosh^2\alpha-\theta\cot\theta)}\}.$$

If the lower sign be taken for the radical r is *positive* when $-\pi\leqslant\theta\leqslant\pi$, and r is an even function of θ; as θ increases from 0 to π, r varies* from $e^{-\alpha}$ to $+\infty$.

If the upper sign were taken for the radical, r would vary from e^{α} to $+\infty$ as θ increases from 0 to $\frac{1}{2}\pi$; and when $\frac{1}{2}\pi<\theta<\pi$, r would be negative.

The curves on which $\theta-2r\sin\theta\cosh\alpha+r^2\sin\theta\cos\theta$ vanishes are shewn in Figs. 1 and 2, the latter shewing the special case $\alpha=0$; the dotted portions correspond to the negative values of r.

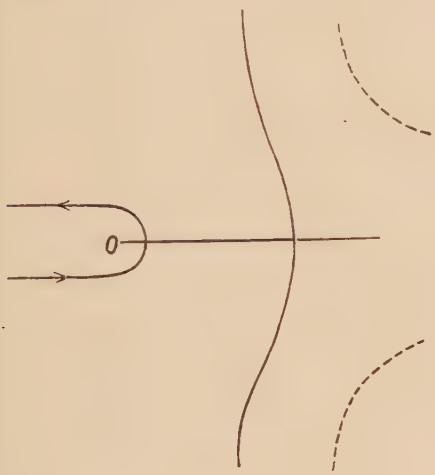

FIG. 1.

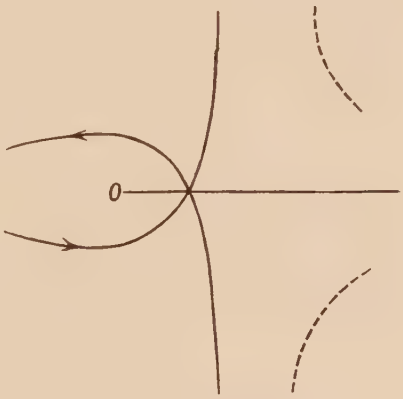

FIG. 2.

Consequently the curve $r=\sec\theta\{\cosh\alpha-\sqrt{(\cosh^2\alpha-\theta\cot\theta)}\}$ is a

* It appears that r does not necessarily increase steadily; for, when $\theta=0$, r is stationary and its second differential coefficient with regard to θ is negative if $\cosh\alpha<2/\sqrt{3}$.

contour of the appropriate type; and it gives rise to the definite integral

$$\Delta_n(\xi) = \frac{1}{2i}\int_{-\pi}^{\pi} e^{-nT}\left(\frac{1}{r}\frac{dr}{d\theta}+i\right)d\theta,$$

where r is the stated function of θ and T is defined as a function* of θ by the equation

$$T = \cosh^2 a+\tfrac{1}{2}+\log r-2r\cos\theta\cosh a+\tfrac{1}{2}r^2\cos 2\theta.$$

To indicate the dependence of T on θ and a, we shall frequently write $T(\theta, \cosh a)$ in place of T; we thus have

$$T(\theta,\cosh a) = \log\left[\sec\theta\left\{\cosh a-\sqrt{(\cosh^2 a-\theta\cot\theta)}\right\}\right]+\cosh^2 a+\tfrac{1}{2}-\theta\cot 2\theta$$
$$-\cosh a\sec^2\theta\left\{\cosh a-\sqrt{(\cosh^2 a-\theta\cot\theta)}\right\}.$$

Since r and T are *even* functions of θ, we see at once that

$$\Delta_n(\xi) = \int_0^{\pi} e^{-nT}d\theta.$$

From the equations

$$dT/d\theta = \{(1/r)-2\cos\theta\cosh a+r\cos 2\theta\}(dr/d\theta)+2r\sin\theta(\cosh a-r\cos\theta),$$
$$0 = 2\sin\theta(r\cos\theta-\cosh a)(dr/d\theta)+(1-2r\cos\theta\cosh a+r^2\cos 2\theta)$$

(the latter being derived from the fact that $I(\tau)$ is zero on the contour), we see that

$$dT/d\theta = r\left[2\sin\theta(\cosh a-r\cos\theta)\right]^{-1}\left[\left\{1/r-2\cos\theta\cosh a+r\cos 2\theta\right\}^2\right.$$
$$\left.+4\sin^2\theta(\cosh a-r\cos\theta)^2\right]$$
$$= r\left[2\sin\theta(\cosh a-r\cos\theta)\right]^{-1}\left[\left\{1/r-2\cos\theta\cosh a+r\right\}^2\right.$$
$$\left.+4\sinh^2 a\sin^2\theta\right].$$

Since $$\cosh a-r\cos\theta = +\sqrt{(\cosh^2 a-\theta\cot\theta)} \geqslant 0,$$

we deduce that, when $0\leqslant\theta\leqslant\pi$, T is a non-decreasing function of θ, and so

$$T \geqslant T(0, \cosh a) = \sinh a\cosh a-a \geqslant 0.$$

It now follows immediately from the last integral given for $\Delta_n(\xi)$ that

$$0 \leqslant \Delta_n(\cosh a) \leqslant \pi e^{-nT(0,\cosh a)}.$$

An inequality for $\Delta_n(\cosh a)$, which is stronger than that just obtained

* In fact $T = R(\tau)$; although τ reduces to T on the contour considered, it avoids confusion to make a distinction between τ, a function of the complex variable s, and T, a function of the real variable θ.

(except when α is small) can be derived from the former of the pair of expressions given for $dT/d\theta$; for we have

$$\begin{aligned} dT/d\theta &\geqslant 2r \sin\theta (\cosh\alpha - r\cos\theta) \\ &= 2\theta \left\{ 1 - \frac{\cosh\alpha}{\cosh\alpha + \sqrt{(\cosh^2\alpha - \theta\cot\theta)}} \right\} \\ &\geqslant 2\theta \{1 - e^{-\alpha}\cosh\alpha\}, \end{aligned}$$

since $\theta\cot\theta \leqslant 1$. Hence, on integration, $T - T(0, \cosh\alpha) \geqslant \frac{1}{2}\theta^2(1-e^{-2\alpha})$, and so

$$\begin{aligned} \Delta_n(\cosh\alpha) &\leqslant \int_0^\pi \exp\{-nT(0, \cosh\alpha) - \tfrac{1}{2}n\theta^2(1-e^{-2\alpha})\}\, d\theta \\ &\leqslant (\tfrac{1}{2}\pi/n)^{\frac{1}{2}}(1-e^{-2\alpha})^{-\frac{1}{2}} \exp\{-nT(0, \cosh\alpha)\}; \end{aligned}$$

and this is the inequality to which reference was made.

The results of this section may be summed up in the following theorem :—

When $x \geqslant 2\sqrt{n}$, $D_n(x)$ is positive and does not exceed the smaller of the functions

$$\Gamma(n+1)\, e^{\frac{1}{2}n}(2n)^{-n}\{x + \sqrt{(x^2-4n)}\}^n \exp\{-\tfrac{1}{4}x\sqrt{(x^2-4n)}\},$$

$$(2\pi)^{-\frac{1}{2}}\Gamma(n+1)e^{\frac{1}{2}n}(2n)^{-n-\frac{1}{2}}(x^2-4n)^{-\frac{1}{4}} \times \{x+\sqrt{(x^2-4n)}\}^{n+\frac{1}{2}} \exp\{-\tfrac{1}{4}x\sqrt{(x^2-4n)}\}.$$

4. *The behaviour of $\Delta_n(\xi)$ and $\Delta_n'(\xi)$ when $\xi \geqslant 1$.*

It is easy to see that

$$\frac{dT(\theta, \xi)}{d\xi} = \frac{\sec^2\theta}{\sqrt{(\cosh^2\alpha - \theta\cot\theta)}} [\{\cosh\alpha\sin^2\theta - \sqrt{(\cosh^2\alpha - \theta\cot\theta)}\}^2 + \cos^2\theta\{\cosh^2\alpha\sin^2\theta + \sinh^2\alpha\}] \geqslant 0;$$

and consequently

$$\frac{dT(\theta, \xi)}{d\xi} = \frac{d}{d\theta}[2\tan\theta\sqrt{(\cosh^2\alpha - \theta\cot\theta)} + 2\cosh\alpha(\theta - \tan\theta)].$$

From the integrals

$$\Delta_n(\xi) = \int_0^\pi e^{-nT}d\theta, \qquad \Delta_n'(\xi) = -n\int_0^\pi \frac{dT(\theta, \xi)}{d\xi} e^{-nT}d\theta,$$

it is now evident (i) that, when $x \geqslant 2\sqrt{n}$, $D_n(x)$ is a positive decreasing

function of x when n is fixed, and (ii) that, when ξ is fixed, and not less than 1, $\Delta_n(\xi)$ and $-n^{-1}\Delta_n'(\xi)$ are positive decreasing functions of n, since

$$T \geqslant T(0, \xi) = \sinh \alpha \cosh \alpha - \alpha \geqslant 0.$$

5. *Simple properties of* $\Delta_n(1)$ *and* $\Delta_n'(1)$.

The functions $\Delta_n(1)$, $\Delta_n'(1)$ are not so amenable to analysis as $J_n(n)$, $J_n'(n)$. It is easily verified that, when θ is small, $27T(\theta, 1)/\theta^3 \sim 8\sqrt{3} - 12\theta$, and so $T(\theta, 1)/\theta^3$ initially *decreases*, though it ultimately *increases*, since it tends to infinity as $\theta \to \pi$.

From a theorem of Bromwich (see his *Infinite Series*, p. 444) it follows [since $T(\theta, 1)$ is monotonic] that

$$\lim_{n\to\infty}[n^{\frac{1}{3}}\Delta_n(1)] = \lim_{n\to\infty}\left[n^{\frac{1}{3}}\int_0^\infty \exp(-8n\sqrt{3}.\theta^3/27)\,d\theta\right] = \tfrac{1}{2}\Gamma(\tfrac{1}{3})/3^{\frac{1}{6}}.$$

Also, it is easy to shew that

$$\frac{d[n^{\frac{1}{3}}\Delta_n(1)]}{dn} = \tfrac{1}{3}n^{\frac{1}{3}}\int_0^\pi \left[\theta\frac{dT}{d\theta} - 3T\right]e^{-nT}\,d\theta;$$

but, since the integrand is not one-signed, we cannot infer that $n^{\frac{1}{3}}\Delta_n(1)$ is monotonic. On the other hand, since the integrand is negative for sufficiently small values of θ, we *can* infer that $n^{\frac{1}{3}}\Delta_n(1)$ is a decreasing function of n *when* n *is sufficiently large.*

Similarly, if we put

$$2\tan\theta\sqrt{(\cosh^2\alpha - \theta\cot\theta)} + 2\cosh\alpha(\theta - \tan\theta) \equiv G(\theta, \cosh\alpha),$$

it follows from § 4 that $\dfrac{d[n^{-\frac{1}{3}}\Delta_n'(1)]}{dn} = n^{\frac{2}{3}}\displaystyle\int_0^\pi (TG' - \tfrac{2}{3}GT')\,e^{-nT}\,d\theta,$

where G is written in place of $G(\theta, 1)$. Since, when θ is small,

$$TG' - \tfrac{2}{3}GT' \sim -8\theta^6/135,$$

it follows that $-n^{-\frac{1}{3}}\Delta_n'(1)$ is a positive increasing function of n, *when* n *is sufficiently large.*

[*Added, December 15th*, 1917.—A proof will be given at the end of § 7*a* that $TG' - \frac{2}{3}GT'$ is negative throughout the range $0 < \theta < \pi$. This result obviously shews that $-n^{-\frac{1}{3}}\Delta_n'(1)$ is a positive increasing function of n *for all positive values of* n.]

6. *Properties of the contour introduced in* § 3.

We shall now make a rather elaborate investigation with a view to proving ultimately that $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$ is a non-increasing function of n when ξ is fixed and not less than 1, provided that $\xi - 1$ is not too large. The range of values of ξ for which we shall prove the theorem is, in fact, $1 \leqslant \xi \leqslant 1{\cdot}2$ approximately. The theorem may be true for greater values of ξ, but it is then comparatively unimportant. The fact that it is true for the range of values just specified will be found later (§ 18) to be of vital importance in dealing with the convergence of series of Kapteyn's type when $\xi \geqslant 1$.

We shall retain the notation of § 3 and consider values of θ between 0 and π only.

Since $$d(r\sin\theta)/d\theta = \sin\theta\,(dr/d\theta)+r\cos\theta$$
$$= \tfrac{1}{2}(1-r^2)/(\cosh\alpha - r\cos\theta)$$
$$= \tfrac{1}{2}(1-r^2)(\cosh^2\alpha - \theta\cot\theta)^{-\frac{1}{2}},$$

we see that the ordinate of the contour increases with θ so long as $r<1$, and decreases when $r>1$.

Now $r=1$ only when $\theta - 2\cosh\alpha\sin\theta+\sin\theta\cos\theta = 0$.

Since* $\theta\operatorname{cosec}\theta+\cos\theta$ decreases steadily from 2 to $\frac{1}{2}\pi$ as θ increases from 0 to $\frac{1}{2}\pi$, and then increases steadily from $\frac{1}{2}\pi$ to $+\infty$ as θ increases from $\frac{1}{2}\pi$ to π, we see that there is *one* value only of θ (other than $\theta=0$), say $\theta=\theta_\alpha$, for which $\theta\operatorname{cosec}\theta+\cos\theta-2\cosh\alpha$ vanishes; and θ_α is an obtuse angle exceeding θ_0, where θ_0 is the value of θ for which $\theta\operatorname{cosec}\theta+\cos\theta-2$ vanishes.

[The angle θ_0 is 2·139 radians; the sexagesimal measure of this angle is 122° 32′.]

It is evident that† $r<1$ when $0<\theta<\theta_\alpha$, and $r>1$ when $\theta_\alpha<\theta<\pi$; and so $r\sin\theta$ increases steadily as θ increases from 0 to θ_α, and then decreases steadily as θ increases from θ_α to π.

We shall subsequently (§ 7) require an upper bound to the value of $f(\alpha)$, where $f(\alpha)$ is the value of $r\sin\theta/(\cosh\alpha - r\cos\theta)$ when $\theta=\theta_\alpha$.

We have

$$f(\alpha) = \sin\theta_\alpha/(\cosh\alpha-\cos\theta_\alpha) = 2\sin^2\theta_\alpha/(\theta_\alpha-\sin\theta_\alpha\cos\theta_\alpha).$$

But $\sin^2\theta_\alpha \leqslant \sin^2\theta_0$, $\theta_\alpha-\sin\theta_\alpha\cos\theta_\alpha \geqslant \theta_0-\sin\theta_0\cos\theta_0$, since θ_α and θ_0 are obtuse angles of which the former is the greater.

Hence $$f(\alpha)\leqslant f(0) = \sin\theta_0/(1-\cos\theta_0) = \cot\tfrac{1}{2}\theta_0 = 0{\cdot}548.$$

That is to say, 0·548 is the upper bound of $f(\alpha)$. The fact which will be found essential in § 7 is that this is less than unity.

We shall next shew that, when $0<\theta<\theta_\alpha$, the contour is *concave* to the initial line.

It is easily seen that

$$\frac{d(r\sin\theta)}{d(r\cos\theta)} = \frac{(1-r^2)\sin\theta}{\cos\theta-2r\cosh\alpha+r^2\cos\theta};$$

and the contour will be *concave*, when $0<\theta<\theta_\alpha$, if this expression

* The differential coefficient of this function is $-\cos\theta\operatorname{cosec}^2\theta\,(\theta-\sin\theta\cos\theta)$.

† If $\alpha=0$, $r=1$ when $\theta=0$; for positive values of α, $r<1$ when $\theta=0$

has a positive differential coefficient with regard to θ for these values of θ.

We shall need the following preliminary results (valid when θ lies between 0 and π) to prove this theorem :—

(i) $1-2r\cos\theta\cosh\alpha+r^2\cos^2\theta = 1-\theta\cot\theta \geqslant 0$,

(ii) $\cos\theta-2r\cosh\alpha+r^2\cos\theta = \cos\theta-\theta\operatorname{cosec}\theta \leqslant 0$,

(iii) $\cosh\alpha-2r\cos\theta+r^2\cosh\alpha \geqslant 1-2r\cos\theta\cosh\alpha+r^2$,

(iv) $2(\sin\theta-\theta\cos\theta) \geqslant \theta-\sin\theta\cos\theta$.

Proofs of (i) and (ii) are left to the reader. To prove (iii), we see that the difference of the expressions is $(\cosh\alpha-1)(1+2r\cos\theta+r^2)$, and this is never negative. To prove (iv), we observe that

$$2(\sin\theta-\theta\cos\theta)-(\theta-\sin\theta\cos\theta)$$

vanishes when $\theta = 0$, while for values of θ between 0 and π it has the positive differential coefficient $2\sin\theta(\theta-\sin\theta)$, and so is not negative.

Now it is easy to shew that

$$\begin{aligned}
&(\cosh\alpha-r\cos\theta)(\cos\theta-2r\cosh\alpha+r^2\cos\theta)^2\frac{d}{d\theta}\left[\frac{d(r\sin\theta)}{d\theta}\Big/\frac{d(r\cos\theta)}{d\theta}\right]\\
&\quad= 2(\cosh\alpha-2r\cos\theta+r^2\cosh\alpha)(1-2r\cosh\alpha\cos\theta+r^2\cos^2\theta)\\
&\qquad\qquad +r(\cos\theta-2r\cosh\alpha+r^2\cos\theta)(1-2r\cos\theta\cosh\alpha+r^2)\\
&\quad\geqslant 2(1-\theta\cot\theta)(1-2r\cos\theta\cosh\alpha+r^2)\\
&\qquad\qquad -r(\theta\operatorname{cosec}\theta-\cos\theta)(1-2r\cos\theta\cosh\alpha+r^2)\\
&\quad= \{2(1-\theta\cot\theta)-r(\theta\operatorname{cosec}\theta-\cos\theta)\}(1-2r\cos\theta\cosh\alpha+r^2)\\
&\quad\geqslant (1-r)(\theta\operatorname{cosec}\theta-\cos\theta)(1-2r\cos\theta\cosh\alpha+r^2) \geqslant 0,
\end{aligned}$$

provided that $r < 1$. Hence the differential coefficient of

$$\{d(r\sin\theta)/d(r\cos\theta)\}$$

is positive when $0 < \theta < \theta_\alpha$, and so *the contour is concave to the initial line for this range of values of* θ.

7. *A family of bicircular quartics associated with the contour.*

It will next be shewn that

$$\frac{dT(\theta,\cosh\alpha)}{d\theta}\Big/\frac{dG(\theta,\cosh\alpha)}{d\theta}$$

is an increasing function of θ, when $0 < \theta < \pi$; where, with the notation of § 5,

$$G(\theta, \cosh a) \equiv 2 \tan\theta \sqrt{(\cosh^2 a - \theta \cot\theta)} + 2 \cosh a\,(\theta - \tan\theta).$$

From the formulæ of §§ 3, 4 it is evident that

$$\frac{dT(\theta, \cosh a)}{d\theta} \Big/ \frac{dG(\theta, \cosh a)}{d\theta} = \frac{(1 - 2r\cos\theta\cosh a + r^2)^2 + 4r^2 \sinh^2 a \sin^2\theta}{2r \sin\theta\,(r^2 - 2r\cos\theta\cosh a + \cosh 2a)}.$$

Now write*

$$X = a\,(\cosh a - r\cos\theta)/\sinh a, \qquad Y = ar\sin\theta/\sinh a,$$

where a is any positive constant, and we have

$$\frac{dT(\theta, \cosh a)}{d\theta} \Big/ \frac{dG(\theta, \cosh a)}{d\theta} = \frac{\sinh a}{2a} \frac{(X^2 + Y^2 - a^2)^2 + 4a^2 Y^2}{Y(X^2 + Y^2 + a^2)}.$$

Hence, if we write

$$(X^2 + Y^2 - a^2)^2 + 4a^2 Y^2 = 4a\lambda Y (X^2 + Y^2 + a^2),$$

we have to shew that λ increases steadily as the point (X, Y) traces out the curve C defined by

$$X = a\,(\cosh a - r\cos\theta)/\sinh a, \qquad Y = ar\sin\theta/\sinh a;$$

where θ is a parameter increasing from 0 to π, and r is, of course, written for brevity in place of $\sec\theta\,\{\cosh a - \sqrt{(\cosh^2 a - \theta\cot\theta)}\}$.

We shall consequently study the family of bicircular quartics

$$(X^2 + Y^2 - a^2)^2 + 4a^2 Y^2 = 4a\lambda Y (X^2 + Y^2 + a^2),$$

λ being the parameter of the family. Only one quartic can be drawn through any real point, other than the points for which both $Y = 0$ and $X^2 + Y^2 = a^2$; and all the quartics pass through these points, which are $(\pm a, 0)$. We shall shew that, as a point P describes the curve C from $\theta = +0$ to $\theta = \pi$, the value of λ for the quartic through P steadily increases from 0 to ∞; and this will establish the theorem to be proved.

Each quartic is symmetrical with regard to the Y-axis, and it is wholly above the X-axis if† $\lambda \geqslant 0$; it meets the Y-axis at $(0, \pm ai)$, $[0, 2\lambda a \pm a(4\lambda^2 - 1)^{\frac{1}{2}}]$, and the two latter points are real only when $\lambda \geqslant \frac{1}{2}$.

* The analysis breaks down when a is actually zero, but it can then be replaced by the simpler analysis of § 7*a*.

† When $\lambda < 0$ the quartic lies below the X-axis, and therefore we may confine our attention to *positive* (including zero) values of λ.

When $\lambda = 0$, the quartic consists of the two points $(\pm a, 0)$; all the quartics touch the axis of X at these points, while the shape of the quartics near them is that of the parabolas $2a\lambda Y = (X \mp a)^2$; these parabolas increase in size with λ.

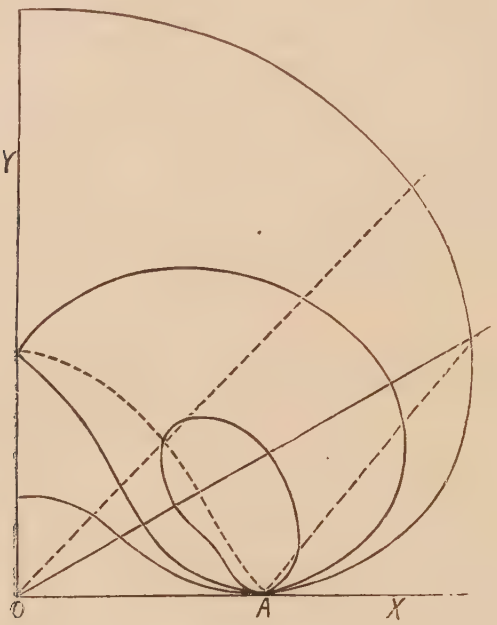

FIG. 3.

Each quartic consists of a number of closed branches,* and this number cannot exceed 2; for if there were 3, a circle drawn through a point on each branch would cut each branch a second time and would meet the curve twice at each circular point, so that a quartic and a conic would have 10 intersections, which is impossible. When $\lambda > \frac{1}{2}$, there cannot even be two branches, for, by symmetry, they would have 0 or 4 intersections with the axis of Y.

Hence, when $\lambda < \frac{1}{2}$, each quartic is bipartite, and, when $\lambda > \frac{1}{2}$, each quartic is unipartite; and by a consideration of the curves near $(\pm a, 0)$, we see that the quartic (λ) is wholly enclosed by the part or parts of the quartic (λ') when $\lambda' > \lambda$.

* The curves shewn in the figure are those for which $\lambda = 1/\sqrt{8}, \frac{1}{2}, 1/\sqrt{2}$; the dotted curve and line are the sextic and its asymptote which will be introduced presently; the line $Y = X \cot \frac{1}{2}\theta_0$ is also shewn.

When $\lambda = \frac{1}{2}$, the quartic has a double point at $(0, a)$, and we have a transition from the bipartite to the unipartite type.

We shall next consider the inflexions of each of the quartics; and we shall discuss them by means of the obvious result that a closed curve cannot have one inflexion without having a second, and if it has two (or more) inflexions, a bitangent can be drawn such that two inflexions lie between its points of contact.

Now the quartic has the following* eight bitangents :—

(i) $Y = 0$, $Y = 2a\lambda/(1-\lambda^2)$.

(ii) $\lambda^2 Y^2 + X^2 = 0$.

(iii) $Y = mX + c$, where

$$c = a\{1 \pm \sqrt{(1-4\lambda^2)}\}/(2\lambda), \qquad m^2 = \{c(1-\lambda^2) - a\lambda\}/(a\lambda^3).$$

The pair (ii) are always imaginary. The set (iii) are real if, and only if, $\lambda \leqslant \frac{1}{2}$, *i.e.*, if the quartic is bipartite; since

$$a^2 - (c/m)^2 = ac\lambda > 0,$$

the set (iii) all pass between the two branches of the curve. Also $Y = 0$ touches at $(\pm a, 0)$, and $Y = 2a\lambda/(1-\lambda^2)$ touches at real points only if $\lambda \leqslant 1/\sqrt{3}$; these points of contact are given by

$$a^2 - X^2 = 4a^2\lambda^4/(1-\lambda^2)^2,$$

and so they lie between the lines $X = \pm a$.

Now consider the area which forms the greatest possible region bounded by arcs of any particular quartic, and the segment of contact of the bitangent $Y = 0$ and the segment of contact of the bitangent

$$Y = 2a\lambda/(1-\lambda^2),$$

the latter being omitted if it does not touch at real points. The boundary of this area is a *simple closed curve* with no real bitangents, except one (or two) which touches at every point of a segment of a straight line. This curve consequently has no inflexions, and since the whole of the portion of the quartic lying on the right of $X = a$ forms part of it, it follows that *no quartic has any inflexions on the right of the line* $X = a$.

---

* Salmon, *Higher Plane Curves*, pp. 231–237, shews that a bicircular quartic has not more than 8 bitangents, and gives a method for deducing them in pairs from four canonical forms of the equation of the quartic; as it does not seem easy to obtain all of the four forms, I derived the bitangents by the direct process of substituting $mX + c$ for Y in the equation, and writing down the conditions that the result should be a perfect square in X.

Hence, if a point starts from $(a, 0)$ to the right and travels along the quartic until it again reaches the line $X = a$, the tangent to the quartic at the point has a continually increasing slope, which increases from 0 to $+\infty$ and then (as the point moves to the left on the upper part of the quartic) from $-\infty$ to a finite negative value.

Moreover, the locus of points of contact with the quartics of tangents parallel to the Y-axis is the sextic

$$Y^6+Y^4(X^2+a^2)-Y^2(X^4-10X^2a^2+a^4)-(X^2-a^2)^2(X^2+a^2)=0.$$

Since the product of the roots of this cubic in Y^2 is positive, and the sum of the roots is negative, the cubic has one positive root and cannot have more than one; and hence, corresponding to any positive value of X, there is one and only one point on the sextic in the *positive quadrant*. The ordinate of this point vanishes when, and only when, $X = a$.

The portion of the sextic in the region in which $Y \geqslant 0$, $X \geqslant a$ consequently consists of a single branch starting from $(a, 0)$ and terminating at infinity* on $Y = X$; and, at points between this branch and the X-axis, the quartics obviously have positive slopes, and are consequently *convex* to the X-axis.

Further, when X is sufficiently large (in fact when $X \geqslant a \times 1{\cdot}9\ldots$ approximately), the sextic lies *above* the line $Y = X\cot\frac{1}{2}\theta_0$, since

$$\cot\tfrac{1}{2}\theta_0 = 0{\cdot}548 < 1,$$

and the asymptote of the sextic is $Y = X$.

Next consider the curve C (introduced on p. 125) traced out by the point whose coordinates are

$$X = a(\cosh a - r\cos\theta)/\sinh a, \qquad Y = ar\sin\theta/\sinh a,$$

as θ increases from 0 to π.

As θ increases from 0 to θ_a the point moves to the right from $(a, 0)$ with an increasing ordinate. This portion of the curve traced out by the point has a decreasing slope, and consequently it cannot meet any particular quartic more than once, since each quartic has an increasing slope on the right of $X = a$ and an increasing ordinate.

The portion of C for which $\theta_a < \theta < \pi$ has a negative slope, and it lies on the right of the line $X = a(\cosh a - \cos\theta_a)/\sinh a$ and below the line $Y = X\cot\frac{1}{2}\theta_0$, by § 6; and so it lies entirely in a region in which the quartics have positive slopes, if we choose a so that

$$a(\cosh a - \cos\theta_a)/\sinh a \geqslant a \times 1{\cdot}9\ldots.$$

* The lines $Y = \pm X$ are the only real asymptotes of the sextic.

Since θ_α is obtuse, this condition is certainly satisfied when $\coth \alpha \geqslant 1{\cdot}9\ldots$, *i.e.*, when $\cosh \alpha \leqslant 1{\cdot}2\ldots$.

When this condition is satisfied, the portion of C under consideration obviously cannot meet any particular quartic more than once.

Hence, if a point P travels along C from $\theta = +0$, the parameter λ_P of the quartic through P must be monotonic as θ increases from 0 to θ_α, and it must also continue to be monotonic as θ increases from θ_α to π; for, if λ_P had any maxima or minima, we could find two points on the curve (one on each side of the critical point) at which λ_P had the same value, and these two points would lie on the same quartic.

Since $\lambda_P \to 0$ as $\theta \to +0$ and $\lambda_P \to \infty$ as $\theta \to \pi$, while λ_P is positive when $\theta = \theta_\alpha$, it is obvious that λ_P must be a steadily increasing function of θ; and so we have established the required result that, when α has any fixed value such that $1 < \cosh \alpha \leqslant 1{\cdot}2\ldots$,

$$\frac{dT(\theta, \cosh \alpha)}{d\theta} \Big/ \frac{dG(\theta, \cosh \alpha)}{d\theta}$$

is an increasing function of θ.

7a. The simplification produced by making $\alpha = 0$.

In the special case in which $\alpha = 0$, the analysis of § 7 breaks down; we then have to prove that $(1-2r\cos\theta+r^2)/(2r\sin\theta)$ is an increasing function of θ.

Write $X = 1-r\cos\theta$, $Y = r\sin\theta$, and we see that we have to prove that $(X^2+Y^2)/(2Y)$ is an increasing function of θ.

We therefore study the family of circles $X^2+Y^2 = 2\lambda Y$. These touch the axis of X at the origin, and have positive slope when $\lambda > 0$, $X > 0$, $Y \leqslant X$. The curve traced out by $(1-r\cos\theta,\ r\sin\theta)$, as θ increases from $+0$ to θ_0, is concave to the axis of X, and the ordinate and abscissa of a point on it are positive and increase as θ, the parameter of the point, increases. This part of the curve consequently meets each circle only once. The portion of the curve for which $\theta_0 < \theta < \pi$ has a negative slope, and lies below $Y = X\cot\frac{1}{2}\theta_0$, and therefore in a region in which the circles have positive slope. The proof that $(X^2+Y^2)/(2Y)$ increases as the point (X, Y) moves along the curve now follows exactly as in § 7.

It may be remarked that, even in this particular case, a direct algebraical proof, that

$$(1-2r\cos\theta+r^2)/(2r\sin\theta)$$

is an increasing function of θ on the curve under consideration, seems to be so arduous as to be impracticable.

[*Added, December 15th*, 1917.—The result stated at the end of § 5, that $TG' - \frac{2}{3}GT'$ is negative when $0 < \theta < \pi$, may be proved by similar graphical considerations: here T, G stand for $T(\theta, 1)$, $G(\theta, 1)$ respectively. For we have

$$\frac{GG'^2}{T'^2} = \frac{8XY^3}{(X^2+Y^2)^2} = 8\sin^3\phi\cos\phi,$$

where $\tan\phi = Y/X$. Now, since $X = 1-r\cos\theta$, $Y = r\sin\theta$, ϕ steadily decreases as θ increases from 0 to π; for when θ is less than θ_0 the curve traced out by (X, Y) is concave to the X-axis, and when $\theta > \theta_0$ the curve has an ordinate which decreases as θ increases. Hence

ϕ steadily decreases from $\frac{1}{3}\pi$ to 0 as θ increases from 0 to π. But, when $0 < \phi < \frac{1}{3}\pi$,

$$\frac{d}{d\phi}(\sin^3\phi\cos\phi) = \sin^4\phi\,(3\cot^2\phi - 1) > 0,$$

and so GG'^2/T'^2 is a steadily decreasing function of θ; hence

$$T'(2GG'' + G'^2) - 2GG'T'' < 0,$$

that is to say,

$$\frac{d}{d\theta}[3T - 2(GT'/G')] < 0.$$

Therefore $3T - 2(GT'/G')$ is a decreasing function of θ which vanishes when $\theta = 0$, and so is negative when $\theta > 0$; and from this result it is obvious that $3TG' - 2GT'$ is negative when $\theta > 0$.]

8. *The cardinal theorem concerning* $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$.

We are now in a position to prove the important theorem that, when ξ is fixed (and $1 \leqslant \xi \leqslant 1{\cdot}2\ldots$), $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$ is a non-increasing function of n.

For the values of ξ under consideration, we have, from the results of §§ 7 and 7a,

$$T''G' - T'G'' \geqslant 0.$$

It follows that

$$\frac{d(T'G/G')}{d\theta} - \frac{dT}{d\theta} = \frac{(T''G' - T'G'')\,G}{G'^2} \geqslant 0,$$

since G is not negative; for G is bounded* when $0 \leqslant \theta \leqslant \pi$, while G vanishes when $\theta = 0$ and $G' = dT/d\xi \geqslant 0$, by § 4, when $0 \leqslant \theta \leqslant \pi$.

Hence $(T'G/G') - T$ is a non-decreasing function of θ, and it is equal to $-T(0, \cosh\alpha)$ when $\theta = 0$; and hence we at once get

$$G(\theta, \cosh\alpha)\frac{dT(\theta, \cosh\alpha)}{d\theta} \geqslant \{T(\theta, \cosh\alpha) - T(0, \cosh\alpha)\}\frac{dG(\theta, \cosh\alpha)}{d\theta}.$$

Next, we shall shew that

$$\Delta_n(\xi)\frac{\partial^2\Delta_n(\xi)}{\partial n\,\partial\xi} - \frac{\partial\Delta_n(\xi)}{\partial\xi}\frac{\partial\Delta_n(\xi)}{\partial n} \leqslant 0.$$

We have

$$\Delta_n(\xi) = \int_0^\pi e^{-nT(\psi,\xi)}\,d\psi, \qquad \frac{\partial\Delta_n(\xi)}{\partial\xi} = -n\int_0^\pi \frac{dG(\theta,\xi)}{d\theta}\,e^{-nT(\theta,\xi)}\,d\theta,$$

and, differentiating under the integral sign, and then integrating by parts,†

* It is easy to see that $G(\theta, \cosh\alpha) = 2(\theta\cosh\alpha - r\sin\theta)$.

† Compare the corresponding analysis in the *Proceedings*, Ser. 2, Vol. 16, pp. 164, 165.

we easily find that

$$\Delta_n(\xi)\frac{\partial^2\Delta_n(\xi)}{\partial n\,\partial\xi}-\frac{\partial\Delta_n(\xi)}{\partial\xi}\frac{\partial\Delta_n(\xi)}{\partial n}=-n\int_0^\pi\int_0^\pi\Omega(\theta,\psi)e^{-n[T(\theta,\xi)+T(\psi,\xi)]}\,d\theta\,d\psi,$$

where
$$\Omega(\theta,\psi)=G(\theta,\xi)\frac{dT(\theta,\xi)}{d\theta}+\{T(\psi,\xi)-T(\theta,\xi)\}\frac{dG(\theta,\xi)}{d\theta}$$

$$\geqslant G(\theta,\xi)\frac{dT(\theta,\xi)}{d\theta}+\{T(0,\xi)-T(\theta,\xi)\}\frac{dG(\theta,\xi)}{d\theta}$$

$$\geqslant 0,$$

by the result just proved; and so we have

$$\Delta_n(\xi)\frac{\partial^2\Delta_n(\xi)}{\partial n\,\partial\xi}-\frac{\partial\Delta_n(\xi)}{\partial\xi}\frac{\partial\Delta_n(\xi)}{\partial n}\leqslant 0,$$

when $1\leqslant\xi\leqslant 1{\cdot}2\ldots.$

It follows from this inequality, exactly as in *Proceedings*, Ser. 2, Vol. 16, pp. 164, 165, that $\Delta_n(\xi)/\Delta_n(1)$ is a non-increasing function of n when ξ is fixed and $1\leqslant\xi\leqslant 1{\cdot}2\ldots.$ Changing to parabolic cylinder functions, we see at once that $D_n(2\xi\sqrt{n})/D_n(2\sqrt{n})$ is a non-increasing function of n when ξ is fixed and $1\leqslant\xi\leqslant 1{\cdot}2\ldots.$

9. *An upper bound to* $|D_n(z)|$ *when the order is a positive integer and* z *is unrestricted.*

It will now be shewn that, when n is an integer,

$$|D_n(z)|<\Gamma(n+1)\,e^{\frac{1}{4}n}(2n)^{-n}\left|\{z+\sqrt{(z^2-4n)}\}^n\exp\{-\tfrac{1}{4}z\sqrt{(z^2-4n)}\}\right|,$$

where that value of $\sqrt{(z^2-4n)}$ is taken which makes

$$z-\sqrt{(z^2-4n)}\leqslant 2\sqrt{n}.$$

In the formula
$$\Delta_n(\zeta)=\frac{1}{2i}\int_{-\infty}^{(0+)}e^{-n\tau}s^{-1}ds,$$

obtained in § 2, take the contour to be a circle, with centre at the origin, terminated at the points at which $\arg s=\pm\pi$, together with the lines joining the end points of this arc to $-\infty$. The integrals along the lines cancel, since n is an integer.

To obtain the best possible inequality, take the circle to pass through a stationary point of τ, namely, the point at which $s=e^{-(\alpha+i\beta)}\equiv s_0$, say, where $\zeta=\cosh(\alpha+i\beta)$ and $\alpha\geqslant 0$, $-\pi\leqslant\beta\leqslant\pi$.

Writing $s=e^{-\alpha+i\theta}$ on the circle and putting $\tau=\tau_0$ when $s=s_0$, we

see that, when s is on the circle,

$$R(\tau-\tau_0) = 2e^{-2a}\sin^2\tfrac{1}{2}(\theta+\beta)\{e^{2a}-\cos(\theta-\beta)\} \geqslant 0.$$

Hence the modulus of $\Delta_n(\zeta)$ cannot exceed $\pi\,|\exp(-n\tau_0)|$.

Now $$s_0 = \cosh(a+i\beta)-\sinh(a+i\beta) = \zeta-\sqrt{(\zeta^2-1)},$$

where that sign is given to the radical which makes $|s_0|<1$; and on substituting for τ_0 in terms of ζ, and thence of z, we at once obtain the inequality stated.

Since $$R(\tau_0) = \tfrac{1}{2}\sinh 2a\cos 2\beta - a \geqslant 0,$$

if $a=0$ or if $\beta=0$, we see that, if z is positive and equal to x (whether greater than, less than, or equal to $2\sqrt{n}$) then $\Delta_n(\xi)<\pi$, and so

$$|D_n(x)| < \Gamma(n+1)\,e^{\frac{1}{4}n}\,n^{-\frac{1}{2}n}.$$

This inequality shews that Adamoff's function $w_n(x)$ is, in reality, $O(n)$ at most, where the constant involved in the symbol O is independent of x.

It is easy to construct a very similar inequality when n is not restricted to be an integer by obtaining a suitable upper bound for the integrand on the straight lines which pass from the circle to infinity.

10. *The expression of a definite integral in terms of parabolic cylinder functions.*

We shall subsequently need the result

$$\int_{-\infty}^{\infty} s^{-n-1}\exp(-\tfrac{1}{4}z^2+n^{\frac{1}{2}}zs-\tfrac{1}{2}ns^2)\,ds = (2\pi n^n)^{\frac{1}{2}}\,e^{\mp\frac{1}{2}(n+1)\pi i}D_{-n-1}(\mp iz).$$

The upper or lower sign is to be taken according as the path of integration passes above or below the origin.

This is most simply proved by taking $R(n)<0$, deforming the contour into the rays $\arg s = 0$, $\arg s = \pm\pi$, expanding $\exp(n^{\frac{1}{2}}zs)$ in powers of z and integrating term by term; the result follows for all values of n by the general theory of analytic continuation.

11. *Asymptotic expansions of $D_n(z)$ when z and n are large.*

We shall now investigate asymptotic expansions corresponding to those obtained by Debye for $J_n(z)$. The analysis is similar to that given by Debye, and will be stated very briefly, except in the discussion of the

range of validity of the expansions; the portions of Debye's work which correspond to the last mentioned investigation seem to be somewhat inadequate.

The expansions will be deduced from the following Lemma:—

LEMMA.—*If* (i) $f(t)$ *is analytic when* $|t| \leqslant a+\delta$, *where* $a>0$, $\delta>0$, *except at a branch-point at the origin, and*

$$f(t) = \sum_{m=1}^{\infty} a_m t^{(m/r)-1}$$

when $|t| \leqslant a$; (ii) $|f(t)| < Ke^{bt}$, *where* K *and* b *are independent of* t, *when* t *is positive and* $t \geqslant a$; (iii) $|\arg n| \leqslant \frac{1}{2}\pi - \Delta$, *where* $\Delta > 0$; (iv) $|n|$ *is sufficiently large: then there exists a complete asymptotic expansion given by the formula*

$$\int_0^{\infty} e^{-nt} f(t)\,dt \sim \sum_{m=1}^{\infty} a_m \Gamma(m/r)\, n^{-(m/r)}.$$

The proof of the lemma is easy; for, if M be any fixed integer, we have

$$\left| f(t) - \sum_{m=1}^{M-1} a_m t^{(m/r)-1} \right| < K_1 t^{(M/r)-1} e^{bt}$$

throughout the range of integration, where K_1 is some number independent of t.

Hence $$\int_0^{\infty} e^{-nt} f(t)\,dt = \sum_{m=0}^{M-1} \int_0^{\infty} e^{-nt} a_n t^{(m/r)-1}\,dt + R_M,$$

where $$|R_M| < \int_0^{\infty} |e^{-nt}|\, K_1 t^{(M/r)-1} e^{bt}\,dt$$

$$< K_1 \Gamma(M/r) / \{R(n)-b\}^{M/r},$$

provided that $R(n) > b$, which is the case when $|n|$ is sufficiently large; and since $\{R(n)-b\}^{-1} = O(1/n)$ for the range of values of n under consideration, we have

$$\int_0^{\infty} e^{-nt} f(t)\,dt = \sum_{m=1}^{M-1} a_m \Gamma(m/r)\, n^{-m/r} + O(n^{-M/r}),$$

and so the integral possesses the complete asymptotic expansion, which is of Poincaré's type.

12. *The application of the method of steepest descents to Whittaker's integral.*

We take the form of Whittaker's integral obtained in § 2, namely,

$$\Delta_n(\xi) = \frac{1}{2i}\int_{-\infty}^{(0+)} e^{-n\tau} s^{-1}\,ds,$$

where $$\tau = \xi^2 + \tfrac{1}{2} + \log s - 2\xi s + \tfrac{1}{2}s^2;$$

this formula is obviously valid when $|\arg n| \leqslant \frac{1}{2}\pi - \Delta$, where Δ is any positive number. As in § 9, we write $\xi = \cosh(\alpha + i\beta)$, where $\alpha \geqslant 0$, $-\pi \leqslant \beta \leqslant \pi$, and also $s = re^{i\theta}$; further, let $\alpha + i\beta = \gamma$, $e^{-\gamma} = s_0$.

It is evident that τ is stationary when $s = s_0$; let the value of τ at this point be τ_0.

Then
$$\begin{aligned}\tau - \tau_0 = \log r + \alpha - r\{e^{\alpha}\cos(\theta+\beta) + e^{-\alpha}\cos(\theta-\beta)\} \\ + \tfrac{1}{2}r^2\cos 2\theta + 1 + \tfrac{1}{2}e^{-2\alpha}\cos 2\beta \\ + i[\theta + \beta - r\{e^{\alpha}\sin(\theta+\beta) + e^{-\alpha}\sin(\theta-\beta)\} \\ + \tfrac{1}{2}r^2\sin 2\theta - \tfrac{1}{2}e^{-2\alpha}\sin 2\beta].\end{aligned}$$

It is necessary, before we apply the method of steepest descents, to study the form of the curve on which $\tau - \tau_0$ is real, namely

$$r^2\sin\theta\cos\theta - r\{e^{\alpha}\sin(\theta+\beta) + e^{-\alpha}\sin(\theta-\beta)\} + \theta + \beta - e^{-2\alpha}\sin\beta\cos\beta = 0.$$

The curve has already been considered (§§ 3, 6) when $\beta = 0$; and since its equation is unaffected by changing the sign of β, provided that the sign of θ be also changed (so that the curve is reflected in the real axis), it is sufficient to consider in detail values of β such that $0 < \beta \leqslant \pi$; further, since the effect of increasing β by π leaves the equation unaffected, provided that θ be diminished by π (*i.e.*, provided the curve is turned through two right angles), it is really sufficient to consider values of β between 0 and $\frac{1}{2}\pi$.

Suppose therefore that $0 < \beta \leqslant \frac{1}{2}\pi$, and consider the values of r corresponding to various given values of θ.

We have $\theta + \beta - e^{-2\alpha}\sin\beta\cos\beta$ positive when $0 \leqslant \theta \leqslant \pi$, and $e^{\alpha}\sin(\theta+\beta) + e^{-\alpha}\sin(\theta-\beta)$ changes sign once (from + to −) at a value of θ greater than $\frac{1}{2}\pi$.

Hence, when $\frac{1}{2}\pi < \theta < \pi$, there is one positive root of the quadratic

in r; this root is a continuous function of θ and tends to a limit when $\theta \to \frac{1}{2}\pi$; but it tends to $+\infty$ as $\theta \to \pi$ (in the latter case, the negative root tends to a finite limit).

When $0 < \theta < \frac{1}{2}\pi$, there are two distinct positive roots so long as

$$\{e^{\alpha}\sin(\theta+\beta)+e^{-\alpha}\sin(\theta-\beta)\}^2-4\sin\theta\cos\theta\{\theta+\beta-e^{-2\alpha}\sin\beta\cos\beta\}>0.$$

The last expression may be written in the form

$$4\sin\theta\cos\theta[\{\sinh^2\alpha\sin^2(\theta+\beta)/(\sin\theta\cos\theta)\} + \{\sin(\theta+\beta)\cos\beta\sec\theta\}-\theta-\beta].$$

The differential coefficient of the expression in square brackets is

$$\sin(\theta+\beta)\sin(\theta-\beta)\sec^2\theta\{\sinh^2\alpha\,\mathrm{cosec}^2\theta+1\},$$

and so the expression has one minimum in the range of values of θ under consideration, namely $\theta=\beta$; its value is then $\cosh 2\alpha\sin 2\beta-2\beta$.

[The expression $\cosh 2\alpha-2\beta\,\mathrm{cosec}\,2\beta$ is positive* when $\zeta\ (\equiv \xi+i\eta)$ is on the right of the curve shewn in Fig. 4 by a continuous line on the right of the η-axis; the curve has a *point saillant* at $\xi=1,\ \eta=0$.]

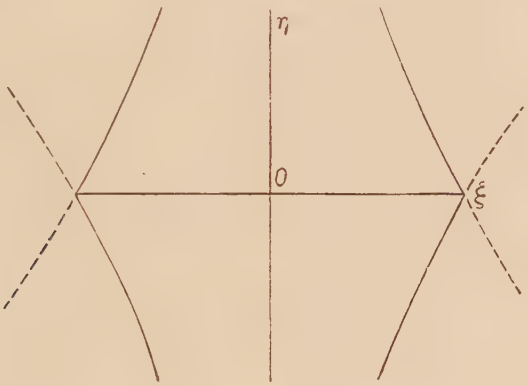

FIG. 4.

There are thus three cases:—(i) If ζ is on the right of the curve, we

* The expression is written in this form to cover the case when β is negative. The reader should note that it did not seem practicable to draw the curves in Figs. 5–11 accurately; and the sketches given indicate merely the general form of the curves.

The dotted curve in Fig. 4 (which is drawn accurately) is $\sinh 2\alpha\cos 2\beta = 2\alpha$; see § 18. The other continuous curve is the image in the η-axis of the first curve; see § 16.

get two distinct values of r for each value of θ between 0 and $\frac{1}{2}\pi$, as in Fig. 5. (ii) If ζ is on the curve, the contour has a second double point

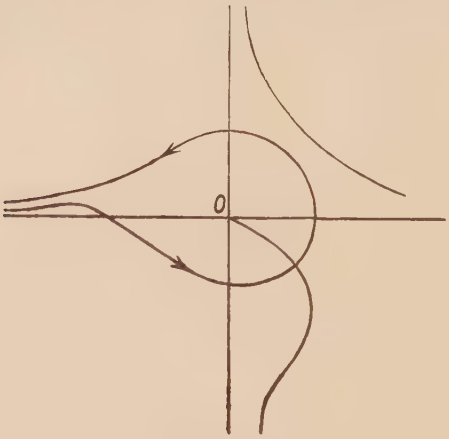

FIG. 5.

where $\theta = \beta$, as in Fig. 6. (iii) If ζ is on the left of the curve, there are two values of θ such that between them r is imaginary, and the curve is as in Fig. 7.

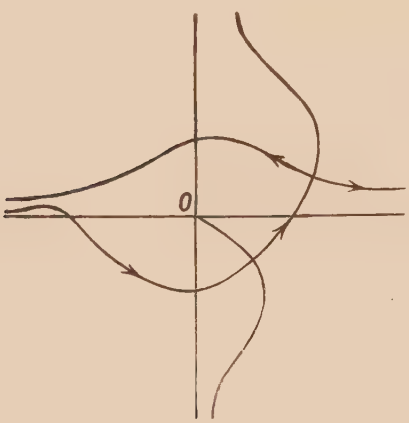

FIG. 6.

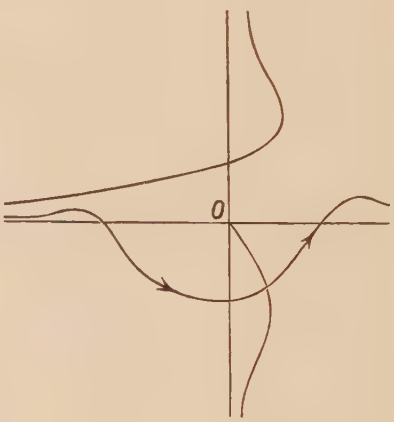

FIG. 7.

Next, when $0 > \theta > -\frac{1}{2}\pi$, one value of r is positive so long as $\theta > -\beta + e^{-2\alpha} \sin\beta \cos\beta$, and when θ is less than this value, both are positive, provided that $e^{\alpha} \sin(\theta+\beta) + e^{-\alpha} \sin(\theta-\beta)$ vanishes for a greater value of θ than $-\beta + e^{-2\alpha} \sin\beta \cos\beta$, *i.e.*, provided that

$$\tan^{-1}(\tanh\alpha \tan\beta) - \beta + e^{-2\alpha} \sin\beta \cos\beta < 0;$$

and this inequality is always true.

[To establish the inequality, we call the expression on the left $f(\alpha, \beta)$; then

$$\frac{\partial f(\alpha, \beta)}{\partial \alpha} = \frac{e^{-2\alpha} \sin\beta \cos\beta (\sinh 2\alpha - \cos 2\beta)}{\cosh^2\alpha \cos^2\beta + \sinh^2\alpha \sin^2\beta}.$$

Hence, if $\cos 2\beta < 0$, $f(\alpha, \beta) \leqslant f(\infty, \beta) = 0$; and if $\cos 2\beta > 0$, $f(\alpha, \beta)$ decreases as α increases from 0 to $\frac{1}{2}\sinh^{-1}(\cos 2\beta)$, and subsequently increases, so that $f(\alpha, \beta)$ does not exceed the greater of $f(0, \beta)$ and $f(\infty, \beta)$. Therefore, always, $f(\alpha, \beta) \leqslant 0$.]

Consequently, when $-\beta + e^{-2\alpha} \sin\beta \cos\beta > \theta > -\frac{1}{2}\pi$, there are two positive values of r which coincide only when $\theta = -\beta$ [the maximum of $\sinh^2\alpha \sin^2(\theta+\beta)/(\sin\theta\cos\theta) + \{\sin(\theta+\beta)\cos\beta \sec\theta\} - \theta - \beta$].

The larger root tends to $+\infty$ as $\theta \to -\frac{1}{2}\pi$, and when $\theta < -\frac{1}{2}\pi$ it becomes negative; the other root remains positive and tends to a finite limit as $\theta \to -\pi$; when $\theta < -\pi$, there are two positive roots (the negative root becoming positive through infinity when $\theta = -\pi$) so long as $\{\sinh^2\alpha \sin^2(\theta+\beta)/(\sin\theta\cos\theta)\} + \{\sin(\theta+\beta)\cos\beta \sec\theta\} - \theta - \beta$ is negative: since the expression has become positive when θ has decreased to $-\pi-\beta$, the curve must lie as shewn in all three figures.

13. *Asymptotic expansions derived from Whittaker's integral.*

We now consider $$\frac{1}{2i}\int e^{-n\tau} s^{-1} ds,$$

taken along the contour marked in that one of Figs. 5, 6, 7 which is appropriate to the particular value of ξ under consideration.

On each contour, τ decreases from $+\infty$ to τ_0, and then increases to $+\infty$. We accordingly write the integral in the form

$$\frac{1}{2i}\int_{\tau_0}^{\infty} e^{-n\tau} \left\{ \frac{d \log(s_1/s_0)}{d\tau} - \frac{d \log(s_2/s_0)}{d\tau} \right\} d\tau,$$

where s_1, s_2 are the two values of s corresponding to a specified value of τ, so chosen that $\arg s_1 \geqslant -\beta \geqslant \arg s_2$.

Now, if we write $v \equiv \log(s/s_0)$, we have

$$\tau - \tau_0 = \sum_{m=0}^{\infty} c_m v^{m+2},$$

where $$c_m = (2^{m+1} s_0^2 - s_0^2 - 1)/(m+2)!.$$

For sufficiently small values of $|\tau - \tau_0|$ this series can be reverted, giving an expansion of the form

$$v = \sum_{m=0}^{\infty} a_m (\tau - \tau_0)^{\frac{1}{2}(m+1)}/(m+1),$$

provided that $c_0 \neq 0$, *i.e.*, provided that $s_0 \neq \pm 1$.

Hence we get $$\frac{dv}{d\tau} = \sum_{m=0}^{\infty} \tfrac{1}{2} a_m (\tau - \tau_0)^{\frac{1}{2}(m-1)}.$$

To determine the radius of convergence of this series, we observe that

$$dv/d\tau = (1 - 2\xi s + s^2)^{-1}.$$

and so v is a monogenic function of τ except when $s = e^{\pm\gamma}$; hence the only possible singularities are at the values of τ given by the formulæ $\tau_0 - 2r\pi i$, $-\tau_0 + 2r\pi i$. Of the latter system the only points which are singularities of the particular branch of v under consideration (defined by the power series and its continuations for positive values of $\tau - \tau_0$) are the three points $-\tau_0$, $-\tau_0 \pm 2\pi i$. For, the singularities being given by $\tau - \tau_0 = 2r\pi i$, $-2\tau_0 + 2r\pi i$, we see that if τ_0 be varied so as to make $-2\tau_0 + 2r\pi i$ tend to zero, then the radius of convergence will tend to zero, and so c_0 must tend to zero; for, if c_0 did not tend to zero, the series would have a non-zero radius of convergence.

Now consider the zeros of $-2\tau_0 + 2r\pi i$, *i.e.*, of $2\gamma - \sinh 2\gamma + 2r\pi i$.

If $$2\gamma - \sinh 2\gamma + 2r\pi i = 0,$$

then $$2r\pi - 2i\gamma = \sin(2r\pi - 2i\gamma).$$

This, regarded as an equation in γ, has a triple zero* when

$$2i\gamma - 2r\pi = 0,$$

and it has simple zeros, two in the strip $\pi < R(2r\pi - 2i\gamma) < \frac{5}{4}\pi$, two in the strip $2\pi < R(2r\pi - 2i\gamma) < \frac{9}{4}\pi$, and so on; none of these latter zeros are on the boundaries of the strips and none are such that

$$I(2r\pi - 2i\gamma) = 0.$$

Hence the only zeros which make c_0 vanish are those for which

$$2i\gamma - 2r\pi = 0;$$

and $\gamma = 0$, $\pm\pi$ are the only ones for which $-\pi \leqslant \beta \leqslant \pi$; these correspond to the values 0, ∓ 1 of r.

Hence the radius of convergence of the series for v is the smallest of 2π, $|2\tau_0|$, $|2\tau_0 \pm 2\pi i|$; and none of these are small except when γ is nearly equal to 0 or $\pm\pi$.

Hence the Lemma of § 11 *is applicable except near the points* $\zeta = \pm 1$, *which will require special consideration.*

---

* These results are due to Hardy, *Messenger of Mathematics*, Vol. 31, p. 163.

Following the methods of Debye,* we find (by using what is practically Lagrange's expansion) that a_m is the coefficient† of v^m in the expansion of $v^{m+1}(\tau-\tau_0)^{-\frac{1}{2}(m+1)}$ in ascending powers of v.

We shall write $$a_{2m}c_0^{m+\frac{1}{2}} \equiv A_m(\gamma),$$

so that

$$A_0(\gamma) = 1,$$

$$A_1(\gamma) = \tfrac{1}{3} - \tfrac{11}{24}\coth\gamma + \tfrac{5}{24}\coth^2\gamma,$$

$$A_2(\gamma) = \tfrac{1}{54} - \tfrac{67}{432}\coth\gamma + \tfrac{427}{1152}\coth^2\gamma - \tfrac{595}{1728}\coth^3\gamma + \tfrac{385}{3456}\coth^4\gamma.$$

In the case of s_1, it is easy to see from Fig. 1 that we must take

$$c_0^{-\frac{1}{2}} = +i/\sqrt{(e^{-\gamma}\sinh\gamma)},$$

where $\sqrt{(\sinh\gamma)}$ is positive when γ is real and positive; for s_2 we must take the other value of $c_0^{-\frac{1}{2}}$, and hence we find that

$$\frac{1}{2i}\int_{\tau_0}^{\infty} e^{-n\tau}\left\{\frac{d\log(s_1/s_0)}{d\tau} - \frac{d\log(s_2/s_0)}{d\tau}\right\}d\tau$$

$$\sim \frac{e^{-n\tau_0}}{2\sqrt{(e^{-\gamma}\sinh\gamma)}}\sum_{m=0}^{\infty}\frac{A_m(\gamma)}{c_0^m}\int_{\tau_0}^{\infty} e^{-n(\tau-\tau_0)}(\tau-\tau_0)^{m-\frac{1}{2}}\,d\tau$$

$$\sim \frac{e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{2\sqrt{(ne^{-\gamma}\sinh\gamma)}}\sum_{m=0}^{\infty}\frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m}.$$

Hence, if γ be such that $\cosh 2\alpha - 2\beta\operatorname{cosec}2\beta > 0$, *i.e., if ξ be on the right of the curve shewn in* Fig. 4, *then the asymptotic expansion of $D_n(z)$ is given by the formula*

$$D_n(z) \sim \frac{\Gamma(n+1)\,e^{\frac{1}{2}n}\,e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{2\sqrt{(\pi n^{n+1}e^{-\gamma}\sinh\gamma)}}\sum_{m=0}^{\infty}\frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m},$$

where $$\gamma = \cosh^{-1}(\tfrac{1}{2}z/\sqrt{n}).$$

The dominant term in this expansion, namely,

$$\tfrac{1}{2}\Gamma(n+1)\exp\{\tfrac{1}{2}n - n\sinh\gamma\cosh\gamma + n\gamma\}/\sqrt{(\pi n^{n+1}e^{-\gamma}\sinh\gamma)},$$

is the second of the two approximations given at the end of § 3 in the special case when z is positive and exceeds $2\sqrt{n}$.]

* In Debye's expression (*Math. Ann.*, Vol. 67, p. 545) a factor 3 is omitted from the coefficient of $c_1^2c_2/c_0^3$ in a_4. The effect of this correction is to change the expression $\frac{7}{576}\cot^2\tau_0$ which occurs in his subsequent work into $\frac{77}{576}\cot^2\tau_0$.

† Expressions for the coefficients a_m as determinants, whose elements are multiples of the coefficients c_r, are given in the *Messenger of Mathematics*, Vol. 46, pp. 97–101.

If, however, ζ is on the left of the curve given by

$$\cosh 2\alpha - 2\beta \operatorname{cosec} 2\beta = 0,$$

then the integral

$$\frac{1}{2i}\int_{\tau_0}^{\infty} e^{-n\tau}\left\{\frac{d\log(s_1/s_0)}{d\tau} - \frac{d\log(s_2/s_0)}{d\tau}\right\}d\tau$$

still has the same asymptotic expansion, *though it represents a different function of* z. For the contour in the s-plane (Fig. 7) is no longer one which is symbolised by $(-\infty; 0+)$; it is a contour going from $\infty e^{-\pi i}$ to $+\infty$ below the origin, or from $+\infty$ to $\infty e^{\pi i}$ above the origin, according* as $I(\zeta) \geqslant 0$ or $I(\zeta) \leqslant 0$, *provided that* ζ *is on the right of the curve*†

$$\cosh 2\alpha - 2(\beta \pm \pi)\operatorname{cosec}(2\beta \pm 2\pi) = 0.$$

Hence, by the result of § 10, the integral is equal to

$$\pm\frac{1}{2i}e^{-\frac{1}{2}n}(2\pi n^n)^{\frac{1}{2}}\, e^{\pm\frac{1}{2}(n-1)\pi i}\, D_{-n-1}(\pm iz),$$

and so we obtain the asymptotic expansion‡

$$D_{-n-1}(\pm iz) \sim \frac{e^{\mp\frac{1}{2}n\pi i}\, e^{\frac{1}{2}n}\, e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{\sqrt{(2n^{n+1}e^{-\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m},$$

where the upper or lower sign is taken according as $I(\zeta) \geqslant 0$ or $I(\zeta) \leqslant 0$, and it is supposed that ζ lies between the curve

$$\cosh 2\alpha - 2\beta \operatorname{cosec} 2\beta = 0,$$

and the curve $\quad \cosh 2\alpha - 2(\beta \pm \pi)\operatorname{cosec}(2\beta \pm 2\pi) = 0.$

---

* If $I(\zeta) = 0$ and $-1 < \zeta < 1$, we may take β positive or negative at pleasure $(-\pi < \beta < \pi)$.

† If ζ were on the left of this curve we should get a contour *starting at* $+\infty$, *encircling the origin, and returning to* $+\infty$. This curve is the image in the imaginary axis of the curve $\cosh 2\alpha = 2\beta \operatorname{cosec} 2\beta$ (see Fig. 4).

‡ When ζ is actually on the curve given by $\cosh 2\alpha = 2\beta \operatorname{cosec} 2\beta$, there is a second stationary point on the contour in the s-plane at which $\tau = -\tau_0$; near this point, $d(\log s)/d\tau$ is $O\{(\tau+\tau_0)^{-\frac{1}{2}}\}$; this pole of the integrand raises the order of the integral along the part of the contour near the point by $n^{\frac{1}{2}}$, which is $o\{\exp(-2n\tau_0)\}$, because τ_0 is not small and $R(\tau_0) < 0$ (since ζ is on the left of the dotted curve in Fig. 4); and so the asymptotic expansion is unaffected when n is positive. If n is complex we may get an additional series to be inserted; this series is easily written down, and it seems unnecessary to give it here. In this case we may go along the contour either to $+\infty$ or to $-\infty$; and so the two functions $D_n(z)$ and $e^{\pm\frac{1}{2}n\pi i}\Gamma(n+1)(2\pi)^{-\frac{1}{2}}D_{-n-1}(\pm iz)$ have the same asymptotic expansion when ζ is on the curve and is not close to the point $\zeta = 1$.

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1918.

Liabilities.	£	s.	d.
Lieut.-Col. Campbell's Fund	499	7	5
Life Compositions Fund	2359	0	0
Invested Surplus Fund	700	0	0
Reserve Fund	34	18	6
Sundry Creditors	236	13	3
De Morgan Medal Fund	2	3	11
Revenue Account balance in hand	95	11	0
	£3927	14	1

Assets.	£	s.	d.
London & North Western Railway £385 Four per Cent. Guaranteed Stock at cost ...	499	7	5
India Stock £700. 7*s.* 1*d.* 3 per Cent. at cost	700	0	0
War Stock £250 5 per Cent. 1929–47 at cost	236	19	5
National War Bonds £2200 5 per Cent., 1st April, 1928, at cost	2200	0	0
Cash at Bank	291	7	3
	£3927	14	1

REVENUE ACCOUNT for the year 1st November, 1917, to 31st October, 1918.

	Year 1916–17					
	£	s.	d.	£	s.	d.
Balance from last year	144	18	6	*107*	*0*	*10*
Income tax returned (2 years)	—			*59*	*9*	*7*
Dividends on—						
Lord Rayleigh's Fund ...	35	1	6	*36*	*4*	*5*
The Society's Investments	104	11	3	*92*	*6*	*4*
Entrance Fees	3	3	0	*1*	*1*	*0*
143 Annual Subscriptions	150	3	0	*128*	*2*	*0*
Sale of *Proceedings*, &c.	118	8	10	*99*	*3*	*9*
	£556	6	1	*523*	*7*	*11*

	Year 1916–17.					
	£	s.	d.	£	s.	d.
Printing and Circulating *Proceedings*, &c.	429	9	6	*328*	*13*	*1*
Purchases for Library	5	9	2	*3*	*7*	*0*
Rent	20	0	0	*41*	*15*	*6*
Postages and Sundries	5	16	5	*4*	*13*	*10*
Balance in hand on 31st October, 1918...	95	11	0	*144*	*18*	*6*
	£556	6	1	*523*	*7*	*11*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.
Balance on 31st October, 1917	9	15	9
Dividends received	2	4	3
	£12	0	0

	£	s.	d.
Paid for De Morgan Medal and Bronze Replica	9	16	1
Balance in hand 31st October, 1918	2	3	11
	£12	0	0

STATEMENT OF TRUST FUNDS, 31st October, 1918.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
Do. do. do. do. £200 Guaranteed 3 per Cent. Stock.

This Fund is held on trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 Five per Cent. Preference Stock.

14th November, 1918

Signed on behalf of the Council:—

H. M. MACDONALD,
A. E. WESTERN.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN J. C. CUNNINGHAM,
Lt.-Col. late R.E., Auditor.

30th November, 1918.

14. *The asymptotic expansion of* $D_{-n-1}(\mp iz)$.

In the preceding section we obtained the asymptotic expansion of $D_{-n-1}(\pm iz)$ when z lay in a certain region of the plane and the upper or lower sign was taken so as to make $R(\pm iz) \leqslant 0$. We shall next consider $D_{-n-1}(\mp iz)$, where $R(\pm iz) \leqslant 0$.

To do this we make use of the second stationary point of

$$\tau \equiv \tfrac{1}{4}\zeta^2 + \tfrac{1}{2} + \log s - 2\zeta s + \tfrac{1}{2}s^2,$$

namely, $s_0' = e^{+(\alpha+i\beta)}$; at this point $\tau = -\tau_0$, and the contour given by the method of steepest descents has for its equation in polar coordinates

$$r^2 \sin\theta \cos\theta - r\{e^{\alpha}\sin(\theta+\beta) + e^{-\alpha}\sin(\theta-\beta)\} + \theta - \beta + e^{2\alpha}\sin\beta\cos\beta = 0,$$

with a double point where $r = e^{\alpha}$, $\theta = \beta$.

When β is positive and acute, the curve is as shewn in Figs. 8, 9, 10, and when β is positive and obtuse, it is as shewn in Fig. 11. The effect of changing the sign of β is to reflect the curve in the real axis. When $\beta = 0$, the figure degenerates into Fig. 1, and when $\beta = \pm\pi$ the figure

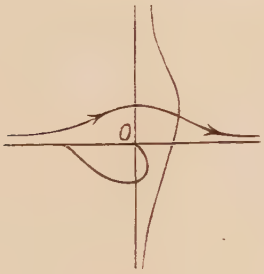

FIG. 8.

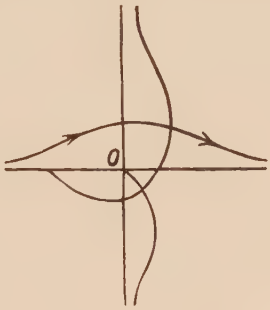

FIG. 9.

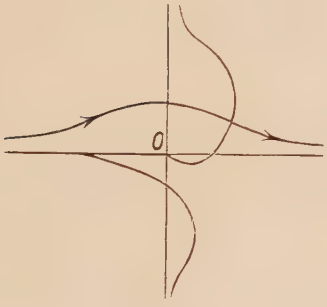

FIG. 10.

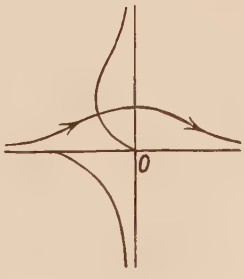

FIG. 11.

also degenerates; in the special case $\beta = 0$, the contour may be taken to

go either from $\infty e^{\pi i}$ or from $\infty e^{-\pi i}$ to $+\infty$, the important stationary point (cf. footnote ‡, p. 140) being that at $-\tau_0$; the cases $\beta = \pm\pi$ are derivable from $\beta = 0$, since $-i\cosh(\alpha \pm i\pi) = +i\cosh\alpha$. The contour going from $\infty e^{\pm\pi i}$ to $+\infty$ gives, in general, the asymptotic expansion of $D_{-n-1}(\mp iz)$.

Writing $w = \log(s/s_0')$, we have

$$\tau+\tau_0 = \sum_{m=0}^{\infty} c_m' w^{m+2},$$

where $$c_n' = \{(2^{n+1}-1)\, s_0^{-2} - 1\}/(n+2)!,$$

and hence $$d(\log s)/d\tau = \Sigma\, \tfrac{1}{2} a_m' (\tau+\tau_0)^{m-\frac{1}{2}},$$

where $a_0' = c_0'^{-\frac{1}{2}}$, $a_1' = -c_1'/c_0'^2$, ..., $(c_0' = e^{\gamma}\sinh\gamma)$, and, generally,

$$a_{2m}' = A_m(-\gamma)/(c_0')^{m+\frac{1}{2}}.$$

To fix the sign of $\sqrt{c_0'}$, we observe that, for the branch of the contour going from s_0' to $+\infty$, $d(\log s)/d\tau$ is *positive* in the limiting case when γ is positive; and so we get, in general,

$$\int_{\infty\exp(\pm\pi i)}^{\infty} e^{-n\tau} s^{-1}\, ds \sim \frac{e^{n\tau_0}}{+\sqrt{(ne^{\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\,\Gamma(m+\frac{1}{2})}{(ne^{\gamma}\sinh\gamma)^m},$$

where $\arg(\sinh\gamma)$ is zero when γ is positive.

From this result* we at once get (except when $\beta = \pm\pi$)

$$D_{-n-1}(\mp iz) \sim \frac{e^{\pm\frac{1}{2}(n+1)\pi i} e^{\frac{1}{2}n} e^{-n(\gamma-\sinh\gamma\cosh\gamma)}}{\sqrt{(2n^{n+1}e^{\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\,\Gamma(m+\frac{1}{2})}{(ne^{\gamma}\sinh\gamma)^m}.$$

[The asymptotic expansion of $D_{-n-1}(\pm iz)$ on the right of the curve $\cosh 2\alpha = 2\beta\operatorname{cosec} 2\beta$ is now at once derivable from the formula

$$D_n(z) = (2\pi)^{-\frac{1}{2}}\Gamma(n+1)\left\{e^{\frac{1}{2}n\pi i} D_{-n-1}(iz) + e^{-\frac{1}{2}n\pi i} D_{-n-1}(-iz)\right\},$$

by a use of the result of § 13.]

Next, write $A_m(\gamma) \equiv A_{m,1}(\gamma) + A_{m,2}(\gamma)$, where $A_{m,1}(\gamma)$, $A_{m,2}(\gamma)$ are respectively even and odd functions of γ. From the formula just quoted, which connects $D_n(z)$, $D_{-n-1}(iz)$, $D_{-n-1}(-iz)$, we see that *between the curves whose equations are*

$$\cosh 2\alpha = 2\beta\operatorname{cosec} 2\beta \quad \text{and} \quad \cosh 2\alpha = 2(\beta \pm \pi)\operatorname{cosec}(2\beta \pm 2\pi),$$

* When $\beta = 0$, there is a second double point on the contour, but this does not affect the result (cf. footnote ‡, p. 140).

$D_n(z)$ possesses the asymptotic expansion

$$D_n(z) \sim \frac{\Gamma(n+1)\, e^{\frac{1}{2}n}\, e^{\pm\frac{1}{4}\pi i}}{\sqrt{(\pi n^{n+1}\sinh\gamma)}}\left[\sum_{m=0}^{\infty}\frac{A_{m,1}(\gamma)\, e^{\frac{1}{2}m\pi i}\,\{\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})\}\cos\phi_m}{(n\sinh\gamma)^m}\right.$$

$$\left. + \sum_{m=1}^{\infty}\frac{A_{m,2}(\gamma)\, e^{\frac{1}{2}(m+1)\pi i}\,\{\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})\}\sin\phi_m}{(n\sinh\gamma)^m}\right],$$

where $$i\phi_m = \tfrac{1}{2}\gamma \mp \tfrac{1}{4}\pi i - n\tau_0 + m\gamma + \tfrac{1}{2}m\pi i,$$

provided that $-\pi < \beta < \pi$, and the upper or lower sign is taken throughout according as $\beta \gtrless 0$.

15. *The connexion of the asymptotic expansion with Adamoff's formula.*

We shall investigate in detail the form assumed by the asymptotic expansion just obtained, in the special case when z is real $(= x)$ and $-2\sqrt{n} < x < 2\sqrt{n}$, n being positive. To interpret $\sqrt{(\sinh\gamma)}$ we suppose that ξ starts by being greater than $+1$, and then describes a semi-circuit about the point $+1$ in the counter-clockwise direction, so that, at a point where $-1 < \xi < 1$, β is positive, α is zero, and $\arg(\sinh\gamma) = +\frac{1}{2}\pi$, $\sinh\gamma$ reducing to $i\sin\beta$; and $x = 2n^{\frac{1}{2}}\cos\beta$, so that

$$\sin\beta = \tfrac{1}{2}\sqrt{\{(4n-x^2)/n\}},$$

while $$\beta = \tfrac{1}{2}\pi - \sin^{-1}(\tfrac{1}{2}x/\sqrt{n}).$$

The upper sign is to be taken in the final asymptotic expansion of § 14, and so we have

$$D_n(x) \sim \frac{\Gamma(n+1)\, e^{\frac{1}{2}n}}{\{1-(x^2/4n)\}^{\frac{1}{4}}\sqrt{(\pi n^{n+1})}}\left[\sum_{m=0}^{\infty}{}'\frac{(-)^m A_{m,1}(i\beta)\,\{\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})\}\cos\phi'_m}{(n^2-\frac{1}{4}nx^2)^m}\right.$$

$$\left. + \sum_{m=1}^{\infty}\frac{(-)^m i A_{m,2}(i\beta)\,\{\Gamma(m+\frac{1}{2})/\Gamma\tfrac{1}{2}\}\sin\phi'_m}{(n^2-\frac{1}{4}nx^2)^m}\right],$$

where $$\phi'_m = \tfrac{1}{4}x\sqrt{(4n-x^2)} + (n-m-\tfrac{1}{2})\sin^{-1}(\tfrac{1}{2}x/\sqrt{n}) - \tfrac{1}{2}n\pi,$$

$$A_{0,1}(i\beta) = 1, \qquad iA_{0,2}(i\beta) = 0,$$

$$A_{1,1}(i\beta) = \tfrac{1}{3} - \tfrac{5}{24}\cot^2\beta, \qquad iA_{1,2}(i\beta) = -\tfrac{11}{24}\cot\beta,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

The dominant term in the expansion of $D_n(x)$ is

$$\Gamma(n+1)e^{\frac{1}{2}n}\,\{1-(x^2/4n)\}^{-\frac{1}{4}}(\pi n^{n+1})^{-\frac{1}{2}}\cos\phi'_0,$$

and, when x is small compared with $n^{\frac{1}{3}}$, we have

$$\phi_0' \sim x\sqrt{n} - \tfrac{1}{2}n\pi,$$

with an error $O(x^3 n^{-\frac{1}{2}})$; from this we deduce the approximate formula

$$D_n(x) \sim \Gamma(n+1)\, e^{\frac{1}{2}n}(\pi n^{n+1})^{-\frac{1}{2}} \cos(x\sqrt{n} - \tfrac{1}{2}n\pi),$$

which is in agreement with Adamoff's approximation.

16. *The asymptotic expansion of $D_n(z)$ in the region hitherto unconsidered.*

In §§ 13 and 14, the asymptotic expansion of $D_n(z)$ was obtained, (i) when ζ is on the right of the curve $\cosh 2\alpha = 2\beta \operatorname{cosec} 2\beta$, and (ii) when ζ is in the region between this curve and its image in the imaginary axis. We now consider the expansion when ζ is on the left of the curve

$$\cosh 2\alpha = 2(\beta \pm \pi) \operatorname{cosec} 2(\beta \pm \pi).$$

We take the formula

$$D_n(z) = e^{\mp n\pi i} D_n(-z) + \frac{\sqrt{(2\pi)}}{\Gamma(-n)} e^{\mp \frac{1}{2}(n+1)\pi i} D_{-n-1}(\pm iz);$$

and we have $$-z = 2n^{\frac{1}{2}} \cosh(\alpha + i\beta \mp i\pi),$$

the upper or lower sign being taken according as β is positive or negative.

Writing $-z$ for z in the formulæ of §§ 13–14, we have

$$D_n(-z) \sim \frac{\Gamma(n+1)\, e^{\frac{1}{2}n} e^{\mp n\pi i} e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{2\sqrt{(\pi n^{n+1} e^{-\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m},$$

$$D_{-n-1}(\pm iz) \sim \frac{e^{\pm\frac{1}{2}(n+1)\pi i} e^{\frac{1}{2}n} e^{\pm n\pi i} e^{-n(\gamma-\sinh\gamma\cosh\gamma)}}{\sqrt{(2n^{n+1} e^{\mp 2\pi i} e^{\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{\gamma}\sinh\gamma)^m}.$$

This gives the result

$$D_n(z) = \frac{\Gamma(n+1)\, e^{\frac{1}{2}n} e^{\mp 2n\pi i} e^{-n(\sinh\gamma\cosh\gamma-\gamma)}}{2\sqrt{(\pi n^{n+1} e^{-\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{(-)^m A_m(\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\gamma}\sinh\gamma)^m}$$

$$+ \frac{\Gamma(n+1)\sin(n\pi)\, e^{\frac{1}{2}n} e^{\pm n\pi i} e^{-n(\gamma-\sinh\gamma\cosh\gamma)}}{\sqrt{(\pi n^{n+1} e^{\gamma}\sinh\gamma)}} \sum_{m=0}^{\infty} \frac{A_m(-\gamma)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{\gamma}\sinh\gamma)^m}.$$

Putting $z = 2n^{\frac{1}{2}} \cosh(\alpha \pm i\pi)$, we get

$$D_n(-2n^{\frac{1}{2}}\cosh\alpha) = \frac{\Gamma(n+1)\, e^{\frac{1}{2}n} e^{\mp n\pi i} e^{-n(\sinh\alpha\cosh\alpha-\alpha)}}{2\sqrt{(\pi n^{n+1} e^{-\alpha}\sinh\alpha)}} \times \sum_{m=0}^{\infty} \frac{(-)^m A_m(\alpha)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{-\alpha}\sinh\alpha)^m}$$

$$- \frac{\Gamma(n+1)\sin(n\pi)\, e^{\frac{1}{2}n} e^{-n(\alpha-\sinh\alpha\cosh\alpha)}}{\sqrt{(\pi n^{n+1} e^{\alpha}\sinh\alpha)}} \sum_{m=0}^{\infty} \frac{A_m(-\alpha)\,\Gamma(m+\frac{1}{2})/\Gamma(\frac{1}{2})}{(ne^{\alpha}\sinh\alpha)^m};$$

This is the form of the result which is useful when the argument of the function is real and negative; and here the positive sign has to be given to each radical. [The reader will remember that, in the former equation, $\arg(e^{\gamma}\sinh\gamma)$ was $\pm 2\pi$].

It will be observed that $\operatorname{cosec}(n\pi)\, D_n(-x)$ is negative when x is sufficiently large and positive, and n is not an integer; this is in agreement with the result given by the ordinary asymptotic expansion.*

17. *Asymptotic expansions connected with $D_n(z)$ when $z \sim 2\sqrt{n}$.*

We shall now obtain the asymptotic expansion of $D_n(z)$ when $\zeta\ (= \frac{1}{2}n^{-\frac{1}{2}}z)$ is nearly equal to one of the two critical values ± 1.

We write $$\tau = \tfrac{3}{2} + \log s - 2s + \tfrac{1}{2}s^2,$$

and consider $$\int e^{-n\tau+\epsilon(s-1)} s^{-1}\, ds,$$

where ϵ is a small number, not restricted to be real, which will be specified precisely later, and the path of integration is one of the following three portions of the contours shewn in Fig. 2: (i) from $s=1$ to $s=+\infty$, (ii) from $s=1$ to $s=\infty e^{\pi i}$, (iii) from $s=1$ to $s=\infty e^{-\pi i}$.

In each case we may write the integral in the form

$$\int_0^{\infty} e^{-n\tau} \exp\{\epsilon(e^v-1)\} \frac{dv}{d\tau}\, d\tau,$$

where $v = \log s$, and so

$$\tau = \tfrac{3}{2} + v - 2e^v + \tfrac{1}{2}e^{2v} = \sum_{m=0}^{\infty} c_m v^{m+3},$$

and $$c_m = (2^{m+2}-2)/(m+3)!\,.$$

We may revert the series expressing τ in terms of v, and when we

* *Proceedings*, Ser. 2, Vol. 8, p. 397.

substitute the result in $\exp\{\epsilon(e^v-1)\}(dv/d\tau)$, we find from Lagrange's expansion that

$$\exp\{\epsilon(e^v-1)\}(dv/d\tau) = \tfrac{1}{3}\sum_{m=0}^{\infty} \tau^{\frac{1}{3}(m-2)} c_0^{-\frac{1}{3}(m+1)} B_m(\epsilon),$$

where $c_0^{-\frac{1}{3}(m+1)}B_m(\epsilon)$ is the coefficient of v^m in the expansion of

$$v^{m+1}(\tfrac{3}{2}+v-2e^v+\tfrac{1}{2}e^{2v})^{-\frac{1}{3}(m+1)}\exp\{\epsilon(e^v-1)\}$$

in ascending powers of v.

The following formulæ give the values of the first few coefficients:

$$B_0(\epsilon)=1, \quad B_1(\epsilon)=-\tfrac{1}{2}+\epsilon, \quad B_2(\epsilon)=\tfrac{17}{80}-\tfrac{1}{4}\epsilon+\tfrac{1}{2}\epsilon^2,$$

$$B_3(\epsilon)=\tfrac{17}{240}+\tfrac{3}{40}\epsilon+0.\epsilon^2+\tfrac{1}{6}\epsilon^3,$$

$$B_4(\epsilon)=-\tfrac{4693}{13440}+\tfrac{1}{6}\epsilon+0.\epsilon^2+\tfrac{1}{24}\epsilon^3+\tfrac{1}{24}\epsilon^4.$$

It is easy to see, from an inspection of Fig. 2, that we must define $c_0^{-\frac{1}{3}}$ for the respective contours to mean $3^{\frac{1}{3}}$, $3^{\frac{1}{3}}e^{\frac{2}{3}\pi i}$, $3^{\frac{1}{3}}e^{-\frac{2}{3}\pi i}$.

Moreover, the expansion in powers of τ defines a function which, with its continuations, is analytic except where v fails to be a monogenic function of τ; since $dv/d\tau = 1/(1-2e^v+e^{2v})$, the singularities of the function are the points where $e^v=1$, *i.e.* where $\tau = 2r\pi i$; the radius of convergence of the expansion is consequently 2π.

Applying the Lemma of § 11, we find that

$$\int_0^{\infty} e^{-n\tau}\exp\{\epsilon(e^v-1)\}\frac{dv}{d\tau}d\tau \sim \tfrac{1}{3}\sum_{m=0}^{\infty} c_0^{-\frac{1}{3}(m+1)}\frac{B_m(\epsilon)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{3}(m+1)}},$$

and a few terms of this series will give a good approximation to the integral, provided that $\epsilon = o(n^{\frac{1}{3}})$.

Now $$D_n(\omega+2\sqrt{n}) = \frac{\Gamma(n+1)e^{\frac{1}{2}n}}{2\pi i n^{\frac{1}{2}n}}\int_{-\infty}^{(0+)} e^{-n\phi(s,\,\omega)}s^{-1}ds,$$

where $$\phi(s,\omega) = (1+\tfrac{1}{2}\omega n^{-\frac{1}{2}})^2+\tfrac{1}{2}+\log s-2(1+\tfrac{1}{2}\omega n^{-\frac{1}{2}})s+\tfrac{1}{2}s^2$$

$$= \tau-n^{-1}\epsilon(s-1)+\tfrac{1}{4}\omega^2 n^{-1},$$

provided that $$\omega = \epsilon n^{-\frac{1}{2}}.$$

From this result, we get at once

$$D_n(\omega+2\sqrt{n}) \sim \frac{\Gamma(n+1)e^{\frac{1}{2}n-\frac{1}{4}\omega^2}}{\pi n^{\frac{1}{2}n}}\sum_{m=0}^{\infty} 3^{\frac{1}{3}(m-2)}\sin\{\tfrac{2}{3}\pi(m+1)\}\frac{B_m(n^{\frac{1}{3}}\omega)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{3}(m+1)}},$$

where ω must be $o(n^{-\frac{1}{6}})$, and $\tfrac{1}{2}\omega/\sqrt{n} = o(n^{-\frac{2}{3}})$.

Similarly, from the formulæ of § 10, we get

$$D_{-n-1}(\pm i\omega \pm 2i\sqrt{n})$$

$$\sim (2\pi n^n)^{-\frac{1}{2}} e^{\mp\frac{1}{2}(n+1)\pi i} e^{\frac{1}{2}n-\frac{1}{4}\omega^2} \sum_{m=0}^{\infty} 3^{\frac{1}{3}(m-2)} \{1 - e^{\pm\frac{2}{3}(m+1)\pi i}\} \frac{B_m(n^{\frac{1}{2}}\omega)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{3}(m+1)}},$$

and hence

$$D_n(-\omega-2\sqrt{n})$$

$$\sim \frac{\Gamma(n+1)e^{\frac{1}{2}n-\frac{1}{4}\omega^2}}{\pi n^{\frac{1}{2}n}}$$

$$\times \sum_{m=0}^{\infty} 3^{\frac{1}{3}(m-2)} \{2\sin\tfrac{1}{3}(m+1)\pi\cos(\tfrac{1}{3}m\pi+\tfrac{1}{3}\pi-n\pi)\} \frac{B_m(n^{\frac{1}{2}}\omega)\,\Gamma(\frac{1}{3}m+\frac{1}{3})}{n^{\frac{1}{3}(m+1)}}.$$

We have now obtained all the results analogous to those obtained by Debye for Bessel functions.

18. *Series of Kapteyn's type.*

The analogue of Kapteyn's type of series of Bessel functions is

$$\Sigma a_n D_n(2\xi\sqrt{n}),$$

where n takes the values ν, $\nu+1$, $\nu+2$, $\nu+3$, ..., and ν is positive, but not necessarily an integer.

From the result of § 8, combined with the formula (cf. § 17)

$$D_n(2\sqrt{n}) = \Gamma(n+1)(e/n)^{\frac{1}{2}n} n^{-\frac{1}{3}} \{\lambda_0+\lambda_1 n^{-\frac{1}{3}}+\lambda_2 n^{-1}+O(n^{-\frac{4}{3}})\},$$

where λ_0, λ_1, λ_2 are independent of n, it follows, exactly as in the corresponding investigation for Bessel functions (*Proceedings*, Ser. 2, Vol. 16, pp. 171–173), that the convergence of the series $\Sigma a_n \Gamma(n+1)(e/n)^{\frac{1}{2}n} n^{-\frac{1}{3}}$ is the necessary and sufficient condition both for the convergence and for the uniform convergence of the series $\Sigma a_n D_n(2\xi\sqrt{n})$ throughout the range $1 \leqslant \xi \leqslant 1{\cdot}2\ldots$. It is also a sufficient condition for the uniform convergence of the same series when $\xi \geqslant 1{\cdot}2\ldots$, by reason of the inequalities of § 3.

19. *The stationary points of $D_n(x)$.*

Since $D_n^2(x)+D_n'^2(x)/(n+\frac{1}{2}-\frac{1}{4}x^2)$ has a positive differential coefficient, namely, $\frac{1}{2}xD_n'^2(x)/(n+\frac{1}{2}-\frac{1}{4}x^2)^2$, when x is positive, it follows as in *Proceedings*, Ser. 2, Vol. 16, p. 169, that the values of $|D_n(x)|$ at its stationary points form an increasing sequence as x takes in turn the critical values between 0 and $2\sqrt{n}$ going from left to right. This

seems a little remarkable when it is considered that $D_n(x)$ is the product of the negative exponential $\exp(-\frac{1}{4}x^2)$ and a polynomial (when n is an integer), but it is quite consistent with the extension of Adamoff's formula given in § 15.

It would be possible, but it seems unnecessary, to obtain limits for the greatest value of $|D_n(x)|$ by methods which I have previously applied to Bessel functions; the results appear to be of no special interest.

20. *The completeness of the system of functions formed by Bessel functions and parabolic cylinder functions.*

The results contained in this paper, together with those given by Debye, complete the discussion of integrals of the type

$$\int s^{-n-1} \exp\{-nAs^m - nBs^p\}\, ds,$$

where m and p are integers (other than zero) by means of the method of steepest descents, when the construction of the appropriate contour involves the solution of only a quadratic equation in s; the case $m = 0$ is reducible to the Eulerian integral for the Gamma function. The memoir published recently by Brillouin contains the simplest results when an equation of degree higher than the second has to be solved, but owing to the logarithmic term present in $\log s + As^m + Bs^p$, the analysis necessary to discuss the integral when m and p have not the values $+1, -1$; $1, 2$; or $-1, -2$, seems, in general, to be extremely elaborate, and it is unlikely that it leads to results of much interest.

[*Added June 15th*, 1918.—I have found that, since my paper was communicated to the Society, a paper by Perron has appeared in the *Munchener Sitzungsberichte*, 1917, pp. 191–220. This paper, which was read in May 1917, deals with the general problem of applying the method of steepest descents to functions represented by contour integrals.]

ON THE SINGULAR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH TRANSCENDENTAL COEFFICIENTS

By Prof. M. J. M. HILL.

[Received and read January 18th, 1917.]

I. *The theory as at present accepted.*

1. If we start from the complete primitive

$$f(x, y, c) = 0, \tag{I}$$

and form the equation

$$\frac{Df}{Dx} + \frac{Df}{Dy}\frac{dy}{dx} = 0, \tag{II}*$$

and then eliminate c between (I) and (II), we obtain a differential equation of the form

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0,$$

or as we shall write it

$$\phi(x, y, p) = 0. \tag{III}$$

Let us now replace c in (I) by a certain function of x, y, viz., C, so that the relation between x, y is

$$f(x, y, C) = 0, \tag{IV}$$

where C satisfies

$$\frac{Df(x, y, C)}{DC} = 0: \tag{V}$$

then on differentiating (IV), and using (V), it follows that

$$\frac{Df(x, y, C)}{Dx} + \frac{Df(x, y, C)}{Dy}\frac{dy}{dx} = 0. \tag{VI}$$

* D denotes partial differentiation when the independent variables are x, y, c or x, y, C or x, y, p; ∂ denotes partial differentiation when the independent variables are x, y; δ denotes partial differentiation when the independent variables are x, c; Δ denotes partial differentiation when the independent variables are x, p; d denotes total differentiation with regard to x; and p denotes $\frac{dy}{dx}$.

Now if C be *not* such a function of x, y as to make

$$\frac{Df(x, y, C)}{Dx} = 0, \qquad \text{(VII)*}$$

and

$$\frac{Df(x, y, C)}{Dy} = 0, \qquad \text{(VIII)*}$$

then it is possible to eliminate C from (IV) and (VI) and the result is again (III).

Hence (IV), on the hypothesis made as to C, is an integral of (III).

It is to be observed that in discussing any particular differential equation, whilst it may be impossible to deduce the singular solution from one form of the primitive (IV) by the use of (V), it may be possible to deduce it from some other form of the complete primitive.

Lagrange supposed that y was expressed in terms of x and c, and instead of using the condition (V) he used the condition here represented by

$$\frac{\delta y}{\delta c} = 0,$$

which is not necessarily equivalent to (V). (See Examples 1, 3, 4.)

The theory of singular solutions as at present accepted assumes that every singular solution of (III) can be obtained in this way, but this is an assumption which is the converse of what has been proved. I have never met with any proof of the converse theorem, nor on the other hand with any particular case in which the converse theorem is not true.

2. If we now turn to the problem of determining the singular solution from the differential equation (III), then the theory as at present accepted states that, if a singular solution of (III) exist, it must satisfy simultaneously (III) and the two following equations

$$\frac{D\phi(x, y, p)}{Dp} = 0, \qquad \text{(IX)}$$

$$\frac{D\phi(x, y, p)}{Dx} + p\frac{D\phi(x, y, p)}{Dy} = 0, \qquad \text{(X)}$$

unless p be infinite, in which case $\frac{D\phi(x, y, p)}{Dy}$ must vanish.

Now it is true that, if a singular solution exists which satisfies (IX),

* If C satisfy (IV) and (V), then (VII) and (VIII) are equivalent to a single condition only.

then it must also satisfy (X), except in the exceptional case noted above. But the theory as at present accepted is an assumption of the converse of this proposition.

Cases are given below (see Examples 1, 3, 4) in which it is impossible to satisfy equation (IX) at all. A case is given (see Example 5) in which the singular solution satisfies (IX), but reduces the left-hand side of (X) to an indeterminate expression.

[2*a*.* Commencing with the primitive in the form taken by Lagrange

$$y = f(x, c), \tag{XI}$$

we obtain the differential equation

$$\frac{dy}{dx} = \frac{\delta f(x, c)}{\delta x}, \tag{XII}$$

or, writing $p = \frac{dy}{dx}$,

$$p = \frac{\delta f(x, c)}{\delta x}. \tag{XIII}$$

Solving (XIII) for c, and substituting in (XI), we obtain an equation of the form

$$y = \chi(x, p). \tag{XIV}$$

I proceed to calculate the partial differential coefficient of y with regard to p, treating x as constant.

Let Δ denote partial differentiation when x and p are treated as independent variables.

Then differentiating (XI) and (XIII) partially with regard to p, we obtain

$$\frac{\Delta y}{\Delta p} = \frac{\delta f(x, c)}{\delta c} \frac{\Delta c}{\Delta p},$$

and

$$1 = \frac{\delta^2 f(x, c)}{\delta c \delta x} \frac{\Delta c}{\Delta p};$$

therefore

$$\frac{\Delta y}{\Delta p} = \frac{\delta f(x, c)}{\delta c} \Big/ \frac{\delta^2 f(x, c)}{\delta c \delta x}. \tag{XV}$$

If now the differential equation be taken in the form

$$\phi(x, y, p) = 0, \tag{III}$$

* This article was added to the paper in February, 1917, subsequently to the date at which the paper was presented to the Society. It is based on Mansion's work, which is referred to in the Historical Section below. The parts so added to the paper are enclosed in square brackets. Mansion's reasoning is differently expressed, but it leads to the same conclusions.

then
$$\frac{D\phi}{Dy}\frac{\Delta y}{\Delta p}+\frac{D\phi}{Dp}=0. \qquad \text{(XVI)}$$

Equating the values of $\frac{\Delta y}{\Delta p}$ in (XV) and (XVI), we obtain

$$\frac{\delta f(x, c)}{\delta c}\Big/\frac{\delta^2 f(x, c)}{\delta c\,\delta x}=-\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy}. \qquad \text{(XVII)}^*$$

Mansion shows that if $\frac{\delta^2 f(x, c)}{\delta c\,\delta x}$ vanishes then each curve represented by the complete primitive has contact of the second order with the envelope. He takes as the condition for finding a singular solution

$$\frac{\Delta y}{\Delta p}=0.$$

It is clear from the equation (XVII) that when

$$\frac{\delta f(x, c)}{\delta c}=0,$$

we must have
$$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy}=0;$$

and therefore it is not sufficient to take

$$\frac{D\phi}{Dp}=0,$$

an equation which it may not be possible to satisfy. It is necessary to consider also the possibility that $\frac{D\phi}{Dy}$ may be infinite.

Mansion did not in his paper apply his condition to any equation having transcendental coefficients. This has been done in Examples 1, 3, 4, 6, 7.]

II. *History of the Subject*, §§ 3–7.

3. The preceding theory, excepting (*a*) the restriction upon C involved in the condition that the equations (VII) and (VIII) are not to be satisfied, and (*b*) the work of Mansion referred to in § 2*a*, dates back to the memoir by Lagrange, "Sur les intégrales particulières des équations différen-

* Mansion does not give this equation.

tielles" (*Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin*, année 1774), printed in the 4th volume of his works.

Lagrange assumes the converse theorems referred to in §§ 1 and 2 above. Thus in § 5, p. 12, of his memoir, after giving the substance of § 1 of this paper, he says:

"Il est facile de démontrer qu'il n'y a pas d'autres combinaisons possibles qui puissent fournir des intégrales de cette espèce non comprises dans l'intégrale complète."

And again, after obtaining the conditions (IX) and (X) (of § 2 of this paper), in § 14, p. 26, on the hypothesis that the differential equation contains no radical signs and no transcendental functions, he says, in § 15, p. 28:

"Et quoique la démonstration précédente soit fondée sur l'hypothèse que l'équation proposée ne renferme aucune fonction transcendante il n'est cependant pas difficile de se convaincre que la même conclusion aura lieu quelles que soient la nature et la forme de cette équation."

It is strange that with this last mentioned idea in his mind Lagrange should have limited his examination of particular cases to equations containing no transcendental functions; and the strangeness is increased by the fact that he refers, *l.c.*, p. 6, to a memoir by Laplace, "Sur les solutions particulières des équations différentielles" (*Mémoires de l'Académie royale des Sciences de Paris*, année 1772), printed in the 8th volume of Laplace's works, in which Laplace compares the differential equations

$$\frac{dy}{dx} = qy^n, \quad \frac{dy}{dx} = q(\log y)^{-r}, \quad \text{and} \quad \frac{dy}{dx} = qe^{-1/y};$$

where q is some function of x, y, which is neither zero nor infinite when $y = 0$, and n and r are both positive (*l.c.*, pp. 330, 331).

4. Darboux in a paper, "Sur les solutions singulières des équations aux dérivées ordinaires du premier ordre", published in the *Bulletin des Sciences Mathématiques*, 1873 (pp. 163, 164), bases his work on the conditions (IX) and (X).

He refers to a paper by Mansion, entitled "Note sur les solutions singulières des équations différentielles du premier ordre" (*Bulletins de l'Académie royale des Sciences, des Lettres et des Beaux-Arts de Belgique*, 1872, pp. 149–169); but he dissents from the view, taken by Mansion,

that the condition

$$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0,^*$$

should be substituted for the condition

$$\frac{D\phi}{Dp} = 0.$$

Darboux says, "Mais ces dernières règles ne s'appliquent qu'aux cas tout à fait singuliers où la fonction ϕ contient des expressions mal déterminées, radicaux, etc.; tout au plus si elles peuvent fournir la solution $p = \infty$, que ne donne pas l'équation

$$\frac{D\phi}{Dp} = 0,$$

et qu'on peut toujours écarter par un changement d'axes coordonnés" (*l.c.*, p. 158, footnote 2).

It will be seen, however, in Examples 1, 3, 4, that their singular solutions are obtainable by Mansion's condition, though not obtainable by Lagrange's condition; and further that they are not merely cases in which $p = \infty$.

In a later memoir, "Sur les solutions singulières des équations aux dérivées partielles du premier ordre", published in 1883 in T. 27 (sér. 2) of the *Mémoires presentés par divers savants à l'Académie des Sciences de l'Institut National de France*, Darboux constructs his theory on equations corresponding to (IX) and (X). These equations he describes as a convention adopted to give a definition of the singular integral applying to all cases (*l.c.*, p. 114). On p. 2, he says that the hypothesis of the existence of a complete integral made by Lagrange was not justified. He says (pp. 2 and 3) that Lagrange supposed the existence of a complete integral which was finite and continuous throughout the whole extent necessary to determine the envelope of the curves represented by the complete integral. He says that this hypothesis excludes, without mentioning it, the case in which the curves represented by the complete primitive have singular points or cease to be continuous or even to exist beyond a limited region of the plane.

Darboux, like Lagrange, does not apply his theory to any particular differential equation with transcendental coefficients. It appears to me to be difficult to see how Darboux's theory, which is based (*l.c.*, pp. 4, 5) on

* Or the equation obtainable from this by the interchange of the dependent and independent variables.

the properties of equations obtained as the result of the elimination of constants, and corresponding closely to the theory of envelopes of algebraic curves and surfaces, could apply without exception to the envelopes of transcendental curves; but if I rightly understand Darboux's reference to "all cases" his theory was intended to include these.

5. Cayley says (*Messenger of Mathematics*, Vol. 6, 1877, p. 24) that the theorem he had proved regarding the existence of an envelope of a family of algebraic curves does not extend to transcendental curves.

6. Hamburger's paper "Ueber die singulären Lösungen der algebraischen Differentialgleichungen erster Ordnung" (*Crelle*, Bd. 112, 1893) is based on the equations (IX) and (X).

But he differs (p. 206) from the opinion expressed by Darboux that the hypothesis made by Lagrange as to the existence of a complete primitive was unjustified, and he shows (pp. 221–229) how the complete primitive is to be found. In this connection he refers not only to the work of Madame Kowalewsky but also to that of Darboux himself.

He also differs from the opinion of Cayley quoted above. His opinion is (pp. 206, 207) that the existence of envelopes or singular solutions has nothing to do with the transcendental or algebraic nature of the general integral.

He points out (p. 238) that inasmuch as the conditions I have marked (VII) and (VIII) [which if taken in conjunction with (IV) and (V) are equivalent to one independent condition] have to be satisfied in order that there may be no envelope, the general case is that in which an envelope exists; whilst, on the other hand, when we start from the differential equation (III), it is necessary that a condition amongst the coefficients [arising from the fact that it must be possible to satisfy simultaneously (III), (IX), and (X)] should be satisfied in order that an envelope may exist, so that the case in which the envelope does not exist is the general one.

7. Laplace, on pp. 327–331 of his memoir, deals with the question of determining whether a given solution of a differential equation is or is not comprised in the general integral, it being supposed that the general integral is unknown. His conclusion is that if the values of the differential coefficients from the second* onwards, as determined from the differential equation, are respectively equal to their values as determined

* The first differential coefficients are necessarily the same.

from the given solution, then that given solution is a particular integral, comprised in the general integral.

If this is not the case, then the solution is a singular integral. This last statement is certainly true, but the one preceding it cannot in all cases be relied on, as will be seen from the discussion of Example 1 below.

Laplace's argument rests on the possibility of expanding the ordinates of points on the complete primitive and on the given solution in ascending powers of the abscissa.

The demonstration therefore may not apply when the points on the given solution are points of discontinuity on the curves represented by the complete primitive.

Laplace's test gives a correct result in Example 2, but fails in Example 1.

It follows from Hamburger's investigations (*l.c.*, p. 219) that the equation considered by Laplace, mentioned above, viz.

$$\frac{dy}{dx} = qy^n,$$

has $y = 0$ for a singular integral when $0 < n < 1$, but a particular integral when $n \geqslant 1$; and thus Laplace's test gives the correct result in this case.

Hamburger's investigations, which are limited to differential equations in which the coefficients are algebraic, do not apply to the other two equations, mentioned by Laplace.

In one of these $$\frac{dy}{dx} = q(\log y)^{-r} \quad (r > 0);$$

therefore $$\frac{d^2y}{dx^2} = \frac{dq}{dx}(\log y)^{-r} - rq^2y^{-1}(\log y)^{-2r-1},$$

so that $$\frac{d^2y}{dx^2} = \infty, \quad \text{when} \quad y = 0.$$

But the solution $y = 0$ makes $\frac{d^2y}{dx^2} = 0$.

Hence $y = 0$ is a singular solution, and Laplace's test gives the correct result.

In the other case $$\frac{dy}{dx} = qe^{-1/y}.$$

Consequently the first and every following differential coefficient tend to zero as y tends to $+0$.

If Laplace's test could be relied on, this would make $y = +0$ a particular integral.

But the equation is not integrable, and I do not know of any method of deciding the question. It should also be observed that, since $e^{-1/y}$ is undefined when $y = 0$, it is probably useless to call $y = 0$, or even $y = +0$, an integral.

I discuss in detail a somewhat similar case: see Example 9 below.

III. *Construction of Differential Equations with Singular Solutions not obtainable from the Differential Equations by the Condition*

$$\frac{D\phi}{Dp} = 0.$$

8. Differential equations with transcendental coefficients have not, I believe, been studied with the object of determining their singular solutions.

As has been said Hamburger's investigations are definitely limited to differential equations with algebraic coefficients.

With regard to Darboux's investigations the passages referred to above would seem to indicate that he held the view that the theory he constructed completely covered all cases of whatever kind.

If, then, the existing theory is incomplete it will be best tested by considering cases in which the differential equations have transcendental coefficients.

Some preliminary theorems which are required are proved in §§ 9–12.

Preliminary Theorems, §§ 9–12.

9. If v be an analytic function of x, y, such that all the values of x, y which make $v = 0$, also make $\dfrac{\partial v}{\partial x} = 0$, then they also make $\dfrac{\partial v}{\partial y} = 0$, and

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2 = 0.$$

For suppose x, y and $x+\delta x$, $y+\delta y$ to be neighbouring points on $v = 0$. Then we must have $v = 0$ at $x+\delta x$, $y+\delta y$; therefore

$$v + \delta x\frac{\partial v}{\partial x} + \delta y\frac{\partial v}{\partial y} + \text{higher powers of } \delta x,\ \delta y = 0.$$

But

$$v = 0, \qquad \frac{\partial v}{\partial x} = 0$$

Hence $$\delta y \frac{\partial v}{\partial y} + \text{higher powers of } \delta x, \delta y = 0.$$

This cannot be satisfied unless $\frac{\partial v}{\partial y} = 0.$

Again, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$, at $x+\delta x$, $y+\delta y$; therefore

$$\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\delta x + \frac{\partial^2 v}{\partial x \partial y}\delta y + \text{higher powers of } \delta x, \delta y = 0$$

and $$\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x \partial y}\delta x + \frac{\partial^2 v}{\partial y^2}\delta y + \text{higher powers of } \delta x, \delta y = 0.$$

These reduce to $$\frac{\partial^2 v}{\partial x^2}\delta x + \frac{\partial^2 v}{\partial x \partial y}\delta y + \ldots = 0,$$

$$\frac{\partial^2 v}{\partial x \partial y}\delta x + \frac{\partial^2 v}{\partial y^2}\delta y + \ldots = 0,$$

and cannot be simultaneously true unless

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 = 0,$$

at all points for which $v = 0$.

If v be a rational integral function of x and y, then the geometrical interpretation of the conditions

$$v = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 = 0,$$

is that every point on the curve represented by $v = 0$ is a cusp. But this is impossible.

The equation $v = 0$ cannot represent a proper curve. Consequently v must break up into factors; and as every point on $v = 0$ satisfies the above conditions, it is necessary that every factor in v should be repeated twice at least.

As examples of functions of v satisfying the above conditions which are not rational, we may take

$$v = (1+w^2)^{\frac{1}{2}} - 1,$$

or $$v = (x^2+y^2+w^2)^{\frac{1}{2}} - (x^2+y^2)^{\frac{1}{2}},$$

where w is a rational function of x and y, and the radicals are all taken with the positive sign.

10. *If v be any analytic function of x, y which does not satisfy the conditions in the preceding article, and u be any function of x, y which is not infinite when $v = 0$, and $\frac{\partial(u, v)}{\partial(x, y)} = 0$ when $v = 0$, then some constant k exists such that $u - k = 0$ when $v = 0$.*

It is supposed, as in § 9, that all the values of x, y which make $v = 0$ do not make $\frac{\partial v}{\partial y} = 0$. Such values define y as a function of x, say

$$y = \psi(x).$$

Then the equation $v(x, y) = 0$ gives $v[x, \psi(x)] \equiv 0$, and therefore

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\psi'(x) = 0,$$

when $y = \psi(x)$. Now we wish to examine the value of u when $v = 0$, *i.e.* $y = \psi(x)$. Then

$$u(x, y) = u[x, \psi(x)] = U(x),$$

say. And now

$$\frac{dU(x)}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\psi'(x),$$

where $y = \psi(x)$. But $\psi'(x) = -\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}$, where $y = \psi(x)$, and therefore

$$\frac{dU(x)}{dx} = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\Big/\frac{\partial v}{\partial y},$$

where $y = \psi(x)$. But $\frac{\partial v}{\partial y} \neq 0$, when $v = 0$, *i.e.* when $y = \psi(x)$; and

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0,$$

when $v = 0$, by hypothesis. Thus $\frac{dU(x)}{dx} = 0$, *i.e.* $u(x, y)$ is a constant, say k, when $v(x, y) = 0$: *i.e.* the integral $v = 0$ is included in the integral $u(x, y) = k$.

The application which will be made of this theorem is the following.

If $u = \text{const.}$ is a complete primitive of (III), and $v = 0$* any solution

* The case where v satisfies the conditions of § 9 is excepted. For if all the values of x, y which make $v = 0$ also make $\frac{\partial v}{\partial x} = 0$, then, as has been shown, they also make $\frac{\partial v}{\partial y} = 0$, and in this case the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ vanishes, but the vanishing of the Jacobian when $v = 0$ implies no relation between u and v. I give two examples (Nos. 6 and 7), in which v is taken to have the form referred to.

of (III) such that $\frac{\partial(u, v)}{\partial(x, y)}$ vanishes when $v = 0$, then a particular integral

$$u - k = 0$$

exists which includes the integral $v = 0$.

11. The converse of the preceding theorem is required.

If the integral of $v = 0$ *be included in the integral* $u = k$, *then*

$$\frac{\partial(u, v)}{\partial(x, y)}$$

will vanish when $v = 0$.

We proceed as in the preceding article as far as the point

$$u(x, y) = u[x, \psi(x)].$$

Then, if $v = 0$ is included in $u = k$,

$$u[x, \psi(x)] = k,$$

and the left-hand side of this equation does not include x at all.

Consequently $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\psi'(x) = 0,$$

where $y = \psi(x)$. But, as in the preceding article,

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\psi'(x) = 0,$$

where $y = \psi(x)$. Hence, since $\frac{\partial v}{\partial y} \neq 0$ when $v = 0$, *i.e.* $y = \psi(x)$, we get, by the elimination of $\psi'(x)$,

$$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

when $v = 0$.*

12. The two preceding theorems make it possible to construct a test for distinguishing a singular solution from a particular solution of a differential equation of the first order, whenever it is possible to express the complete primitive in the form

$$u = k.$$

For, let v be any solution of the differential equation, and then take u in such a form that u is not infinite when $v = 0$, and then form

$$\frac{\partial(u, v)}{\partial(x, y)}.$$

* I have substituted the argument of §§ 10 and 11, which was suggested to me by Mr. G. H. Hardy, for that contained in the original version of the paper.

It follows from the preceding work that if $\frac{\partial(u, v)}{\partial(x, y)}$ vanishes when v vanishes, then v is included in the complete primitive (§ 10); and if v is included in the complete primitive, then $\frac{\partial(u, v)}{\partial(x, y)}$ vanishes when v vanishes (§ 11). This gives the following test.

If $\frac{\partial(u, v)}{\partial(x, y)}$ *do not vanish when* v *vanishes, then the solution* $v = 0$ *is not included in the complete primitive, and therefore it is a singular solution.*

13. It sometimes happens that a solution of a differential equation is both a particular and a singular solution of a differential equation.

E.g., $y = 0$ is both a particular integral and a singular solution of the differential equation

$$p^4 = 4y(xp - 2y)^2,$$

of which the complete primitive is

$$y = c^2(x - c)^2.$$

Hamburger, *l.c.*, p. 218, gives tests by which a solution could be recognised as both singular and particular.

In the example just referred to, the solution $y = 0$ arises in two ways :—

(1) by putting $c = 0$, which shows it is a particular integral;

(2) by putting $c = x$, which shows it is a singular integral, and possesses the envelope property.

In this paper I am concerned solely with those solutions which can only be obtained from the complete primitive by replacing the arbitrary constant by a function of the variables. In special cases, as in the one referred to above, a solution may have a double origin, but these are not considered here.

14. The following primitive has a considerable degree of generality, but it is not quite so general as that discussed in § 19 below, of which it is a particular case.

Consider the equation

$$f(x, y) + \phi(w)\,\psi(x, y) = c, \qquad \text{(XVIII)}$$

where w is some function of x and y which is such that all the values of x, y, which make $w = 0$, do not make $\frac{\partial w}{\partial x} = 0$ or $\frac{\partial w}{\partial y} = 0$; where $\phi(w)$ vanishes when $w = 0$; where $\phi'(w)$ is infinite when $w = 0$; where $f(x, y)$ neither vanishes, nor reduces to a constant, nor becomes infinite when $w = 0$; and where $\psi(x, y)$ neither vanishes nor becomes infinite when $w = 0$.

Differentiating (XVIII), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \phi(w)\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}\right) + \phi'(w)\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y) = 0. \tag{XIX}$$

The last term is indeterminate when $w = 0$, because $\phi'(w)$ is infinite, and

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} = 0.$$

Dividing (XIX) by $\phi'(w)$, the equation can be put into the form

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}\right) + \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y) = 0. \tag{XX}$$

Now $w = 0$ is an integral of (XX).

But $w = 0$ is not a particular case of the complete primitive, for if we put $w = 0$ in the complete primitive, then in order that the equation may be satisfied c must be equal to the value of $f(x, y)$ when $w = 0$. But, by hypothesis, $f(x, y)$ is not constant when $w = 0$. Hence $w = 0$ is not included in the complete primitive, and must therefore be a singular solution.

To see whether $w = 0$ satisfies the condition (IX), we observe that the equation (XX), after putting $\frac{dy}{dx} = p$, becomes

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right) + \frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial y}\right) + \left(\frac{\partial w}{\partial x} + p\frac{\partial w}{\partial y}\right)\psi(x, y) = 0.$$

The condition (IX) is therefore

$$\frac{1}{\phi'(w)}\frac{\partial f}{\partial y} + \frac{\phi(w)}{\phi'(w)}\frac{\partial \psi}{\partial y} + \psi(x, y)\frac{\partial w}{\partial y} = 0,$$

which, when $w = 0$, reduces to

$$\psi(x, y)\frac{\partial w}{\partial y} = 0.$$

But, by hypothesis, neither $\psi(x, y)$ nor $\frac{\partial w}{\partial y}$ vanishes when $w = 0$.

Hence the condition $\frac{D\phi}{Dp} = 0$ cannot be satisfied.

[Let us see whether the condition $\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0$ can be satisfied. If we calculate $\frac{D\phi}{Dy}$, then one of the terms contains the expression

$$\phi''(w) / \{\phi'(w)\}^2.$$

It will be seen in § 16 that, in all the cases which I have been able to construct, this expression becomes infinite when $w = 0$. So that it may be expected that $\frac{D\phi}{Dy}$ will be infinite, and therefore that the condition $\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0$ will be satisfied.]

For the primitive (XVIII), the condition (V) reduces to the equation $-1 = 0$, which cannot be satisfied. But, in all the cases which I have been able to construct, I have found it possible, after transforming the primitive, and taking Lagrange's form of the complete primitive, to derive the singular solution from the equation

$$\frac{\partial y}{\partial c} = 0.$$

15. It will next be shown that the solution $w = 0$ possesses the envelope property.

For writing $u = f(x, y) + \phi(w)\,\psi(x, y)$,

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \phi(w)\frac{\partial \psi}{\partial x} + \psi(x, y)\,\phi'(w)\,\frac{\partial w}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \phi(w)\frac{\partial \psi}{\partial y} + \psi(x, y)\,\phi'(w)\,\frac{\partial w}{\partial y}.$$

Now, when $w = 0$, $\phi'(w) = \infty$.

Hence at a point at which $w = 0$ meets one of the primitives $u = \text{const.}$, we have

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{\partial w}{\partial x} \Big/ \frac{\partial w}{\partial y} \qquad \text{(XXI)}$$

so that, at a point where a complete primitive $u = \text{const.}$ meets $w = 0$, these two curves have the same tangent.

It might at first sight appear likely that it would follow from (XXI) that

$$\frac{\partial(u, w)}{\partial(x, y)}$$

would vanish where $w = 0$, but that is not the case, for

$$\frac{\partial(u, w)}{\partial(x, y)} = \frac{\partial(f, w)}{\partial(x, y)} + \phi(w)\frac{\partial(\psi, w)}{\partial(x, y)}.$$

Hence, when $w = 0$, $$\frac{\partial(u, w)}{\partial(x, y)} = \frac{\partial(f, w)}{\partial(x, y)}.$$

Now, if $\dfrac{\partial(f, w)}{\partial(x, y)} = 0$, when $w = 0$, then, by § 10, a constant k exists such that $f(x, y) = k$ when $w = 0$; but this is contrary to the hypothesis in § 14 regarding $f(x, y)$.

Hence $\dfrac{\partial(u, w)}{\partial(x, y)}$ does not vanish when $w = 0$; and therefore $w = 0$ is a singular solution, according to the test of § 12.

16. Let us now consider the conditions $\phi(w) = 0$, and $\phi'(w) = \infty$, when $w = 0$.

(i) If $\phi(w)$ be an algebraic function of w, these conditions require that $\phi(w)$ must contain a factor w^n, where $0 < n < 1$.

Let us take* $$\phi(w) = w^n.$$

Then the equation (XVIII) becomes

$$w^n = \{c - f(x, y)\}/\psi(x, y);$$

therefore $$w = \{c - f(x, y)\}^{1/n}/\{\psi(x, y)\}^{1/n}.$$

If now we differentiate the primitive in this form with regard to c, then, since $1/n$ is greater than 1, the resulting equation can be satisfied by giving to c the value $f(x, y)$, leading to the singular solution $w = 0$.

Thus the primitive can in this case be transformed into one in which the ordinary method for deducing the singular solution from the primitive is applicable.

* The remaining factors of $\phi(w)$ may be included in $\psi(x, y)$.

The cases $$\phi(w) = \sin(w^n) \quad (0 < n < 1),$$

and $$\phi(w) = \tan(w^n) \quad (0 < n < 1),$$

are not essentially different from this case.

(ii) Taking next the case

$$\phi(w) = (\log w)^{-r} \quad (r > 0),$$

then $$\phi'(w) = -r(\log w)^{-r-1} w^{-1}.$$

In this case the equation (XVIII) becomes

$$(\log w)^{-r} = \{c - f(x, y)\}/\psi(x, y).$$

Therefore $$\log w = \{\psi(x, y)\}^{1/r}/\{c - f(x, y)\}^{1/r};$$

and therefore $$w = \exp\left[\{\psi(x, y)\}^{1/r}/\{c - f(x, y)\}^{1/r}\right].$$

If we differentiate the primitive in this form with regard to c, the condition to be satisfied is that the exponential last written, multiplied by

$$-\frac{1}{r}\{\psi(x, y)\}^{1/r}\{c - f(x, y)\}^{-1/r-1},$$

shall tend to zero as c is made to approach $f(x, y)$.

Examples are given below where this is the case: see Examples 1, 4, 6.

In these examples only one curve of the family of complete primitives passes through each point of the plane. Cases of exception may arise in consequence of an indeterminacy (see Example 7).

(iii) Consider next the case,

$$\phi(w) = w(\log w)^n \quad (n > 0).$$

Here $$\phi'(w) = (\log w)^n + n(\log w)^{n-1}.$$

Taking the primitive in the form

$$w(\log w)^n = \{c - f(x, y)\}/\psi(x, y),$$

it is apparently impossible to satisfy the condition (V). Bearing in mind the remark at the end of § 1, we take in place of (V) the condition

$$\frac{\delta y}{\delta c} = 0.$$

Differentiating the primitive with regard to c, treating x as a constant,

it follows that

$$[(\log w)^n + n(\log w)^{n-1}]\frac{\partial w}{\partial y}\frac{\delta y}{\delta c}$$

$$= \left\{\left(1-\frac{\partial f}{\partial y}\frac{\delta y}{\delta c}\right)\Big/\psi(x, y)\right\} - \{[c-f(x, y)]/[\psi(x, y)]^2\}\frac{\partial \psi}{\partial y}\frac{\delta y}{\delta c}.$$

Therefore

$$\frac{\delta y}{\delta c} = 1\Big/\left[\{(\log w)^n + n(\log w)^{n-1}\}\frac{\partial w}{\partial y}\psi(x, y) + \frac{\partial f}{\partial y} + \{c-f(x, y)\}\psi^{-1}\frac{\partial \psi}{\partial y}\right].$$

Since neither $\frac{\partial w}{\partial y}$ nor $\psi(x, y)$ vanish when w vanishes, the great square bracket becomes infinite when $w = 0$; and therefore the condition $\frac{\delta y}{\delta c} = 0$ leads to the singular solution $w = 0$ (see Example 3).

Cases of the same kind can be constructed by taking

$$\phi(w) = w(\log w)\{\log(\log w)\}^n,$$

$$\phi(w) = w(\log w)\{\log(\log w)\}[\log\{\log(\log w)\}]^n, \ \ldots,$$

where $n > 0$.

In cases (ii) and (iii) the locus $w = 0$ is such that, as a point on any one of the curves represented by the complete primitive approaches $w = 0$, the tangent line to the curve tends to coincide with the tangent line to $w = 0$, but the curve usually ends abruptly at the point at which it meets $w = 0$, because if w be negative, $\log w$ is imaginary.

For cases of exception see Examples 6 and 7.

17. A similar phenomenon to that noticed in parts (ii) and (iii) of the preceding article occurs if a certain arbitrary restriction be made, in the case of complete primitives which are algebraic.*

Suppose that, instead of considering the equation

$$p^2x - py + 1 = 0,$$

we consider the equation obtained by solving it for p, but that we make the restriction that the radical shall always be taken with the positive sign, so that we consider

$$p = \{y + (y^2-4x)^{\frac{1}{2}}\}/(2x).$$

---

* Boole, *Finite Differences*, 2nd edition, ch. x, § 22.

Then the integral is $c = \{y+(y^2-4x)^{\frac{1}{2}}\}/(2x)$,

and this represents only that portion of a tangent line to the parabola

$$y^2-4x = 0,$$

which lies on one side of the point of contact.

18. A noteworthy result is obtained by taking the same form for the complete primitive as in § 14, but with a different hypothesis as to the nature of $\phi(w)$.

Suppose that $\phi(w) = e^{-1/w^2}$,

so that $\phi'(w) = 2w^{-3}e^{-1/w^2}$.

In this case both $\phi(w)$ and $\phi'(w)$ tend to zero as w tends to 0.

The differential equation is now

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right)+e^{-1/w^2}\left(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}\right)+2w^{-3}e^{-1/w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x,y)=0. \quad \text{(XXII)}$$

This equation is not satisfied by $w = 0$.

If, however, the equation is written in the form

$$w^3\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right)+w^3e^{-1/w^2}\left(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}\right)$$
$$+2e^{-1/w^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x,y)=0, \quad \text{(XXIII)}$$

then $w = 0$ satisfies this equation.

The question arises whether $w = 0$ is to be regarded as an integral of the equation (XXIII) or not.

Goursat, in his *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, has on p. 35, ll. 8–15, a passage from which it seems to follow that he would regard a solution of this kind as an integral of the equation.

As against this point of view it may be observed—

(i) That $w = 0$ does not satisfy the equation in virtue of the value it gives for dy/dx, for it causes the term independent of dy/dx and the coefficient of dy/dx each to vanish.

(ii) That the curve $w = 0$ does not touch the curves of the complete primitive where it meets them.

In fact the value of dy/dx, at a point on a complete primitive where it meets $w = 0$, is

$$-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y},$$

which is not equal to

$$-\frac{\partial w}{\partial x}\Big/\frac{\partial w}{\partial y}.$$

These are two reasons for not regarding $w = 0$ as an integral of the differential equation (XXIII).

It seems to me that equation (XXIII) should be regarded as resoluble into (XXII) and $w = 0$; and that $w = 0$ ought not to be regarded as an integral of (XXIII).

19. A more general form of complete primitive than the one in § 14 is the following:—

$$f(x, y, c) + \phi(w)\,\psi(x, y, c) = 0, \qquad \text{(XXIV)}$$

where $\phi(w)$ has the same properties as in that section, whilst $f(x, y, c)$ and $\psi(x, y, c)$ only differ from $f(x, y)$ and $\psi(x, y)$ by containing c *algebraically*, and $f(x, y, c)$ does not vanish when w vanishes.

The result of differentiating (XXIV) becomes, after division by $\phi'(w)$,

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\,\frac{dy}{dx}\right) + \frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\,\frac{dy}{dx}\right) + \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\,\frac{dy}{dx}\right)\psi(x, y, c) = 0. \qquad \text{(XXV)}$$

If we find the values of c which satisfy (XXIV), and put each of them in the left-hand side of (XXV), then the result in each case vanishes when $w = 0$, and therefore the differential equation obtained by eliminating c between (XXIV) and (XXV) will be satisfied by $w = 0$.

Now $w = 0$ is not obtainable from (XXIV) by giving to c any particular constant value, since $f(x, y, c)$ does not vanish when we put $w = 0$.

Consequently $w = 0$ is a singular solution of the differential equation of which (XXIV) is the complete primitive. A particular example of this case is Example 5.

EXAMPLE 1.

Illustration of §§ 7, 14, *and* 16 (ii).

Consider the differential equation

$$p = y\,(\log y)^2.$$

The complete primitive can be expressed in either of the forms

$$x+(1/\log y)-c = 0,$$

or

$$y = \exp[-1/(x-c)].$$

It represents the family of curves obtained by moving the curve $y = \exp(-1/x)$ parallel to the axis of x.

This curve has a salient point at the origin, and touches the axis of x there. Hence $y = 0$ is an envelope of the family.

It is not possible to derive the solution $y = 0$ from the complete primitive by giving to c any constant value.

If we put $f(x, y, c) \equiv x+(1/\log y)-c$, we cannot obtain the singular solution from the condition (V); but if we put

$$f(x, y, c) \equiv y-\exp[-1/(x-c)],$$

the condition (V) is satisfied by putting $c = x$. As x tends to $c+0$, y tends to 0.

If we apply the test of § 12, and put

$$u = x+(1/\log y), \qquad v = y,$$

then

$$\frac{\partial(u, v)}{\partial(x, y)} = 1,$$

so that $y = 0$ ranks as a singular solution.

[It should be observed further that as x tends to $c-0$, $\log y$ tends to infinity, and that we may have not only $y = 0$ but $y = \infty$ as a singular solution. For, if we put $y = 1/Y$ and $x = -X$, the differential equation becomes

$$\frac{dY}{dX} = Y(\log Y)^2,$$

so that its form is unaltered. Hence $Y = 0$ is a singular solution of this equation. Consequently $y = \infty$ should rank as a singular solution of the original equation. The fact that $y = \infty$ is a part of the envelope is perhaps more readily seen by transforming the primitive by replacing y by $(y+1)/(y-1)$. See Ex. 1*a*.]

If we put $$\phi(x, y, p) \equiv p - y(\log y)^2,$$

it is impossible to satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

[In this case $$\frac{D\phi}{Dy} = -(\log y)^2 - 2(\log y).$$

Hence the condition $$\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0,$$

leads to the condition that $\log y = \infty$; and therefore $y = 0$ or $y = \infty$, both of which are parts of the envelope.]

In this case Laplace's Test for a Singular Solution fails. All the differential coefficients of y with regard to x as determined from the differential equation vanish when $y = 0$. But $y = 0$ is not a particular integral as required by Laplace's test. The explanation is that the point in which a complete primitive meets the singular integral is a point of discontinuity on the complete primitive, and the expansion which must be possible if Laplace's argument is to apply is here impossible.

EXAMPLE 1*a*.

[Consider the differential equation

$$p + \tfrac{1}{2}(y^2 - 1)(\log v)^2 = 0,$$

where $$v = (y+1)/(y-1).$$

The integral can be expressed in either of the forms

$$x + (1/\log v) = c,$$

or $$y = [e^{1/(c-x)} + 1] / [e^{1/(c-x)} - 1].$$

Now $y = 1$ and $y = -1$ are both integrals of the differential equation. They are singular integrals, as they cannot be obtained from the complete primitive by giving to c any constant value.

The integral $y = 1$ is obtained by making $c - x = +0$, and the integral $y = -1$ is obtained by making $c - x = -0$.

The primitive has two salient points, viz., at $x = c$, $y = \pm 1$.

At each of these points the tangent is parallel to the axis of x. Hence $y = +1$ and $y = -1$ are parts of the envelope.

The condition $\frac{\delta y}{\delta c} = 0$ is satisfied by $c - x = \pm 0$, and so gives the singular integrals.

The condition $\frac{D\phi}{Dp} = 0$ cannot be satisfied, but the condition

$$\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0$$

is satisfied by $y = \pm 1$.]

EXAMPLE 2.

Illustration of § 7.

Consider the equation $p = y \log y$.

The integral is $\log (\log y) = x - c$, if $y > 1$,

but $\log [\log (y^{-1})] = x - c$, if $0 < y < 1$.

If $y < 0$ the integral is imaginary.

Now $y = 0$ is obviously a solution of the differential equation.

Let us examine its relation to the integral

$$\log [\log (y^{-1})] = x - c.$$

This integral can be expressed in the form

$$e^x / [\log (y^{-1})] = \text{const.},$$

and so $y = 0$ is the particular integral given by making the constant vanish.

In this case the second and all higher differential coefficients of y with respect to x vanish when $y = 0$.

Thus Laplace's method furnishes a correct result in this case.

If we put $u = e^x / [\log (y^{-1})], \quad v = y,$

then $$\frac{\partial (u, v)}{\partial (x, y)} = u;$$

and therefore vanishes when $y = 0$. Therefore

$$\frac{\partial (u, v)}{\partial (x, y)} = 0,$$

when $v = 0$, and so the test of § 12 makes $y = 0$ a particular integral.

If we had taken for u the form

$$u = \log[\log(y^{-1})] - x, \quad v = y,$$

we should have found $$\frac{\partial(u, v)}{\partial(x, y)} = -1,$$

and the integral would have appeared to be a singular integral. But with this form of u, the condition that u must be taken in a form which is finite when $v = 0$ is not fulfilled, and therefore the Jacobian has not been calculated in accordance with the directions in § 12.

EXAMPLE 3.

Illustration of §§ 7, 14, *and* 16 (iii).

Consider the equation $p = (1+\log y)^{-1}$.

This is a particular case of the equation noted by Laplace,

$$\frac{dy}{dx} = q(\log y)^{-r},$$

where $r > 0$ and q is neither zero nor infinite when $y = 0$.

The integral is $y \log y - x + c = 0$.

Now $y = 0$ is an integral of the differential equation.

It can only be derived from the complete primitive by putting $c = x$. It is therefore a singular solution.

If we take the primitive in the form

$$f(x, y, c) \equiv y \log y - x + c = 0,$$

we cannot satisfy the condition (V).

If, however, we calculate $\frac{\delta y}{\delta c}$ from the primitive, we get

$$\frac{\delta y}{\delta c} = -(1+\log y)^{-1},$$

and therefore the condition $\frac{\delta y}{\delta c} = 0$ requires $\log y$ to be infinite, and consequently y to be zero or infinite.

The former gives the singular solution; the latter can be regarded as the particular integral corresponding to an infinite value of c.

It is impossible to satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

[But $$\frac{D\phi}{Dy} = y^{-1}(1+\log y)^{-2}.$$

therefore $$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = y(1+\log y)^2.$$

Hence the condition $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$ requires that $y = 0$, and thus gives the singular solution.]

It may be observed that in this case

$$\frac{D\phi}{Dx} + p\frac{D\phi}{Dy} = py^{-1}(1+\log y)^{-2},$$

and is therefore indeterminate when $y = 0$, $p = 0$. It would not be permissible before examining the value of this expression to put

$$p = (1+\log y)^{-1},$$

because this value of p does not arise directly from the solution under consideration, viz., $y = 0$.

EXAMPLE 4.

Illustration of §§ 14 *and* 16 (ii).

Consider the equation

$$p[1+y(\log y)^2] = y(\log y)^2.$$

The complete primitive is

$$x-y+(1/\log y) = c.$$

There is a salient point at $x = c$, $y = 0$. The tangent at this point is the axis of x.

The family of curves is obtained by moving any one of their number parallel to the axis of x.

The curve has two asymptotes, viz.,

$$y = 1, \quad y = x-c.$$

Starting from the salient point, y increases towards 1, while x tends to $+\infty$.

The curve then passes to the other end and to the upper side of the asymptote $y = 1$.

As x increases from $-\infty$ to $+\infty$, y increases from $+1$ to $+\infty$, and the curve approaches the asymptote $y = x - c$.

Since the solution $y = 0$ can only be obtained from the primitive by putting $c = x$, it is a singular solution.

The same result follows from the test of § 12.

It is not possible to satisfy the condition (V) with the primitive in the form taken above.

But if we calculate $\frac{\delta y}{\delta c}$, we get

$$\frac{\delta y}{\delta c} = -y(\log y)^2/[1+y(\log y)^2],$$

so that $\frac{\delta y}{\delta c} = 0$ gives the singular solution $y = 0$.

The solution $y = 0$ will not satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

[But $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = [1+y(\log y)^2]/[(p-1)\{(\log y)^2+2(\log y)\}]$,

so that the condition $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$ leads to $y = 0$, and thus gives the singular solution.]

EXAMPLE 5.

Illustration of § 19.

$$p^2x+py\log y-y^2(\log y)^4 = 0.$$

The complete primitive is

$$x+(c/\log y)-c^2 = 0,$$

and $y = 0$ is an integral of the differential equation.

But it is a singular integral because to obtain it from the complete primitive it is necessary to put $c = x^{\frac{1}{2}}$. So also is $y = \infty$.

The complete primitive can be written

$$y = \exp[c/(c^2-x)].$$

If the curve be transformed to the point $x = c^2$, $y = 0$ as origin, it

becomes

$$Y = \exp(-c/X),$$

a curve of the same nature as that of Example 1.

The curves obtained by taking $c = +a$, $c = -a$ are reflexions of each other in the line $x = a^2$.

Thus in this case the salient point is confined to the positive part of the axis of x.

There is, however, another singular solution obtainable by making the values of c or of p equal. To obtain it we observe that the point in which the curve

$$x+(c/\log y) = c^2$$

meets the curve $$x+(a/\log y) = a^2,$$

is given by $$x = -ac, \quad y = \exp[1/(a+c)].$$

Hence the point at which the curve

$$x+(c/\log y) = c^2$$

meets the envelope is

$$x = -c^2, \quad y = \exp[1/(2c)].$$

The equation of the envelope is

$$1+4x(\log y)^2 = 0.$$

This curve satisfies Lagrange's conditions.

On the other hand the singular integral $y = 0$ will satisfy the conditions

$$\phi = 0, \quad \frac{D\phi}{Dp} = 0.$$

But $$\frac{D\phi}{Dx}+p\frac{D\phi}{Dy} = p^2\log y+2p^2-2py(\log y)^3(\log y+2),$$

which contains the indeterminate term $p^2 \log y$, since $p = 0$, $\log y = \infty$, when $y = 0$.

Both singular integrals can be obtained from the primitive, taken in the form

$$y = \exp[c/(c^2-x)],$$

by using the equations (IV) and (V).

The reason why $\frac{D\phi}{Dp} = 0$ is satisfied in this case is that the two primitives through the point $x = a^2$, $y = 0$, for which $c = +a$ and $c = -a$, meet and touch the same line $y = 0$. In this case $y = 0$ is not only an

envelope, but seems to rank also as a tac-locus, inasmuch as the tangent lines to the two curves tend to coincide with the same line, as their points of contact approach the salient points.

EXAMPLE 6.

Illustration of §§ 9 *and* 16 (ii).

$$(v+2)\,p = y(v+1)(\log v)^2,$$

where $v = (1+y^2)^{\frac{1}{2}}-1.$

The complete primitive is

$$x+(1/\log v) = c.$$

There is a cusp at $x = c$, $y = 0$; the tangent at the cusp being the axis of x.

There are three asymptotes

$$x = c, \quad y = 3^{\frac{1}{2}}, \quad y = -3^{\frac{1}{2}}.$$

The family of curves is formed by moving any one of their number parallel to the axis of x, which touches each of them, so that it is their envelope.

The figure when $c = 0$ is very similar to that which would be obtained by taking the curve

$$y = \exp(-1/x),$$

together with its reflexion in the axis of x, and then multiplying each ordinate by $3^{\frac{1}{2}}$.

The function v is of the kind considered in § 9, and therefore, if we take

$$u = x+(1/\log v),$$

then $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$ when $v = 0$, for both $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ vanish when $v = 0$, and it would seem that $v = 0$ is a particular integral.

But the only values of x and y which make $v = 0$ are given by

$$(1+y^2)^{\frac{1}{2}}-1 = 0,$$

and therefore by

$$y = 0.$$

Suppose therefore we take

$$u = x+[1/\log\{(1+y^2)^{\frac{1}{2}}-1\}], \quad v = y,$$

then $$\frac{\partial(u, v)}{\partial(x, y)} = 1,$$

showing that $y = 0$ is a singular solution.

If the complete primitive be thrown into the form

$$y^2 = 2e^{1/(c-x)} + e^{2/(c-x)},$$

then the singular solution can be obtained from the equations (IV) and (V).

If the differential equation be written in the form

$$p = y(v+1)(\log v)^2/(v+2),$$

$$\frac{D\phi}{Dp} = 1,$$

$$-\frac{D\phi}{Dy} = 2\log v + (\log v)^2 \{(v+1)^2+v\}/\{(v+1)(v+2)\}.$$

Thus the condition $$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$$

requires that $\log v$ should be infinite, and therefore gives the singular solution $y = 0$. And $y = \infty$ is also a singular solution.

EXAMPLE 7.

Illustration of §§ 9 *and* 16 (ii).

$$(v+2)(xp-y) = xy(v+1)(\log v)^2,$$

where $$v = \left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}} - 1.$$

The primitive is $$x + (1/\log v) = c,$$

and can be put into the form

$$y^2 = x^2\{2e^{1/(c-x)} + e^{2/(c-x)}\}.$$

The point $x = c$, $y = 0$ is a cusp on every curve of the family, the tangent at the cusp being the axis of x.

The origin is a double point on every curve of the family (a cusp when $c = 0$).

There are three asymptotes,

$$x = c, \quad y = \sqrt{3}[x-(2/3)], \quad y = -\sqrt{3}[x-(2/3)].$$

Since $y = 0$ makes $p = 0$, and $v = 0$ so that $y(\log v)^2 = 0$, it is an in-

tegral. Moreover as $y = 0$ touches the curve at $x = c$, it is an envelope, and it is also a cusp-locus.

The function v belongs to the class of functions discussed in § 9.

If we take $$u = x + (1/\log v),$$

then $$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

when $v = 0$, because both $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ vanish when $v = 0$; but $v = 0$ is not a particular integral.

When $v = 0$, it is necessary to have $y = 0$ or $x = \infty$. If now we take

$$u = x + \left[1/\log\left\{\left(1 + \frac{y^2}{x^2}\right)^{\frac{1}{2}} - 1\right\}\right],$$

$$v = y,$$

then $$\frac{\partial(u, v)}{\partial(x, y)} = 1 + x^{-1}(v+2)(v+1)^{-1}(\log v)^{-2},$$

which becomes equal to 1 when $v = 0$, and so indicates that $y = 0$ is a singular solution.

In this case, putting

$$\phi(x, y, p) = (v+2)(xp - y) - xy(v+1)(\log v)^2,$$

we get $$\frac{D\phi}{Dp} = x(v+2),$$

which does not vanish when $y = 0$.

[But $$\frac{D\phi}{Dy} = (xyp - y^2)\, x^{-2}(v+1)^{-1} - (v+2) - x(\log v)^2\left(\frac{2v^2+4v+1}{v+1}\right) - 2x(v+2)\log v,$$

which becomes infinite when $y = 0$, so that the condition

$$\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0,$$

gives the singular solution $y = 0$. And $y = \infty$ is also a singular solution.]

EXAMPLE 8.

Illustration of § 16 (i).

$$x\cos(y^r)\,p = y^{1-r} \quad (0<r<1).$$

The primitive is $x-\sin(y^r)=c,$

and can be written $y=[\arcsin(x-c)]^{1/r};$

and since the exponent $1/r$ is greater than 1, the condition $\delta y/\delta c=0$ gives

$$c=x,$$

and therefore leads to the singular solution $y=0.$

The condition $$\frac{D\phi}{Dp}=0$$

requires that $r\cos(y^r)=0$ when $y=0,$ which cannot be satisfied.

[On the other hand the condition $\dfrac{D\phi}{Dp}\Big/\dfrac{D\phi}{Dy}=0$ is satisfied by $y=0.$]

Taking the case $r=\frac{1}{2}$ as the simplest, we have

$$(c-x)^2=\tfrac{1}{2}[1-\cos(2y^{\frac{1}{2}})]=\tfrac{1}{2}\left[\frac{4y}{2!}-\frac{(4y)^2}{4!}+\frac{(4y)^3}{6!}-\ldots\right].$$

Treating this equation as $f(x,y,c)=0,$ then $\dfrac{Df}{Dc}=0$ gives

$$\cos(2y^{\frac{1}{2}})=1;$$

therefore $$y^{\frac{1}{2}}=s\pi,$$

where s is zero or a positive or negative integer.

If $s=0,$ we get the envelope $y=0.$

If we take any other value of s we do not get a solution of the equation at all, but a node-locus.

The equation is, if $r=\frac{1}{2},$

$$\cos(y^{\frac{1}{2}})\,p=2y^{\frac{1}{2}}.$$

Squaring to remove the irrational powers of $y,$ we get

$$\left[2-\frac{1}{2!}(4y)+\frac{1}{4!}(16y^2)-\ldots\right]p^2=8y,$$

If we now make $\frac{D\phi}{Dp} = 0$, we get

$$\left[2 - \frac{1}{2!}(4y) + \frac{1}{4!}(16y^2) - \ldots\right] p = 0.$$

Thus (i) $p = 0$ gives the singular solution $y = 0$. And (ii) we have also

$$2 - \frac{1}{2!}(4y) + \frac{1}{4!}(16y^2) - \ldots = 0,$$

i.e. $$1 + \cos(2y^{\frac{1}{2}}) = 0,$$

whence $$\cos(y^{\frac{1}{2}}) = 0;$$

therefore $$y^{\frac{1}{2}} = (2t+1)\frac{\pi}{2},$$

which gives $p = 0$, but, on putting this value of y in the differential equation, we get

$$p = \infty.$$

In this case the differential equation is not satisfied.

We have, in fact, a tac-locus, the difference between the values of the parameters of two touching curves being 2.

If $r = \frac{1}{2}$, the curve has a simple contact with the envelope.

If $r = \frac{1}{3}$, the envelope coincides with an inflexional tangent to each curve.

If $r = \frac{2}{3}$, the envelope coincides with the tangent at a cusp on each curve. In this case the envelope is at the same time a cusp-locus.

If r be negative, $y = 0$ is still a solution of the differential equation, but in this case the complete primitive has a point of indetermination at the point where it meets the singular solution.

EXAMPLE 9.

Illustration of § 18.

Let $$2p + y^3 e^{-1/y^2} = 0.$$

The integral is $$y^{-2} = \log(c+x).$$

For y to be real, we must have $x \geqslant 1-c$. When $x = 1-c$, $y = \pm\infty$.

If $y > 0$, $p < 0$, and as y tends to $+0$, p tends to -0 and x to $+\infty$. If $y < 0$, $p > 0$, and as y tends to -0, p tends to $+0$ and x to $+\infty$.

The curve is symmetrical with regard to the axis of x. As x increases

from $1-c$ to $+\infty$, y decreases from $+\infty$ to $+0$ for one branch, and increases from $-\infty$ to -0 for the other branch.

One asymptote is $x = 1-c$. The curve tends to opposite ends of the side of this asymptote for which $x > 1-c$.

Another asymptote is $y = 0$. The curve tends to opposite sides of the end of this asymptote for which $x = +\infty$.

$y = 0$ is a solution of the differential equation. It is a particular integral, obtainable by putting $c = +\infty$.

Since
$$y = [\log(c+x)]^{-\frac{1}{2}},$$
$$\frac{\delta y}{\delta c} = -\tfrac{1}{2}[\log(c+x)]^{-\frac{3}{2}}(c+x)^{-1},$$
so that $\dfrac{\delta y}{\delta c} = 0$ when $c = +\infty$.

It might seem therefore that $y = 0$ was a singular solution. But $y = 0$, though it touches all the curves of the system, touches them all at infinity, for $p = 0$ when $y = \pm 0$, $x = +\infty$.

It is not however an envelope in the usual sense, viz., that in which every point of the envelope is the point of contact of one curve of the system.

EXAMPLE 10.

$$p = y^2 e^{-1/y}.$$

The integral is
$$x + e^{1/y} = c.$$

The family of curves is obtained by moving the curve $x + e^{1/y} = 0$ parallel to the axis of x. If we change x into $-Y$, and y into $-X$, we get the curve of Example 1, viz.:—
$$Y = e^{-1/X}.$$

As x passes from $-\infty$ to -1, y increases from $+0$ to $+\infty$, and approaches the upper end of the asymptote $x = -1$. The curve then passes to the other side of the lower end of this asymptote. Then, as x increases from -1 to -0, y increases from $-\infty$ to -0.

The tangent at the origin is the axis of y.

Consequently $y = 0$ is not a singular solution of the differential equation.

But $y = 0$ is an asymptote to every curve of the family.

The question noticed in § 18, as to whether $y = 0$ can be regarded as

a solution of the equation

$$p = y^2 e^{-1/y}$$

at all, presents considerable difficulty.

Since the integral is $x + e^{1/y} = c$,

as y tends to $+0$, x tends to $-\infty$, and p to zero.

The only point, however, at which the curve actually reaches the axis of x is at $x = c$, when y tends to -0, but in this case $p = +\infty$.

To obtain the solution $y = +0$ from the primitive, it is necessary to put $c = +\infty$.

And so, if $y = 0$ is to be regarded as a solution, it must be regarded as a particular integral.

But since the family of curves is obtained by moving the curve

$$x + e^{1/y} = 0$$

parallel to the axis of x, there is no point of that axis, at a finite distance from the origin, which both lies on a complete primitive and is such that the complete primitive touches the axis at it.

The equation considered in this example is the most similar in form to that mentioned by Laplace (viz.

$$\frac{dy}{dx} = qe^{-1/y},$$

where q is neither zero nor infinite when $y = 0$) which I have been able to integrate.

Both in the equation considered by Laplace and in the one here studied, where q is zero when $y = 0$, the values of all the differential coefficients tend to 0 as y tends to $+0$.

As has been seen in the study of Example 1, this does not make it certain that $y = 0$ is a particular integral.

EXAMPLE 11.

$$(y^2 + e^{-1/y})\, p = y^2.$$

The complete primitive is

$$x - y - e^{-1/y} = c.$$

There is a salient point at $x = c$, $y = 0$.

The tangent here is $y = x - c$.

There are two asymptotes $y = 0$ and $y = x - c - 1$.

Starting from the salient point, as x increases to $+\infty$, y increases to $+\infty$, and the curve approaches the upper end of the asymptote

$$y = x - c - 1.$$

The curve then passes to the lower end of this asymptote, and then, as x increases from $-\infty$ to $+\infty$, y increases to -0 and tends to approach the asymptote $y = 0$.

The question noticed in § 18 arises.

Is $y = +0$ to be regarded as an integral?

It satisfies the equation $(y^2 + e^{-1/y})\, p = y^2$,

but does not satisfy the equation

$$(1 + y^{-2} e^{-1/y})\, p = 1.$$

Moreover $y = +0$ is certainly not an envelope, for each member of the complete primitive meets it at an angle of 45°.

My view is that the equation

$$(y^2 + e^{-1/y})\, p = y^2$$

should be regarded as being resoluble into

$$(1 + y^{-2} e^{-1/y})\, p = 1,$$

and $y = 0$.

A PROPERTY OF THE PENCIL FORMED BY THE TANGENTS FROM ANY POINT TO A NODAL CUBIC, AND SOME PROPERTIES OF THE QUADRANGLE FORMED BY THEIR POINTS OF CONTACT

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1. The property in question is that any cross-ratio of the pencil of four tangents is connected by a rational algebraic equation with the corresponding cross-ratio of the pencil formed by joining the points of contact to the node.

The reciprocal theorem is that any cross-ratio of the four points in which a straight line meets a three-cusped quartic is connected by a rational algebraic relation with the corresponding cross-ratio of the four parameters of the points. By the parameter of the point I mean, here and elsewhere, the parameter in terms of which its coordinates can be expressed *rationally*. I have considered the question whether any other unicursal quartic possesses this property, and find that, apart from the trivial case of a quartic with a triple point, there is only one other, namely the quartic with three nodes, each nodal tangent being inflexional. Projectively then the only unicursal quartics, other than a quartic with a triple point, which possess this property, are the cardioid and the lemniscate of Bernoulli.

As this discussion seems to be of more fundamental interest than anything else I have to say, I give it first.

In the rest of the paper I discuss some properties of the points of contact of the tangents from any point to the nodal cubic. I show that there is a conic passing through them distinct from the polar conic of the point, which is noteworthy in some ways, and I show how the constructions of the tangents from any point to the cubic may be effected by the intersection of two conics, each of which may be constructed by ruler and compasses. I show also how, given two points on the cubic, real or conjugate imaginaries, the other two points, such that the tangents at them meet at

the same point as the tangents to the first two points, may be constructed by ruler and compasses. In these cases the cubic is supposed to be determined by its node and six other points.

2. (i) If a quartic has a triple point, which we take to be the origin, then the coordinates of any point on it can be expressed rationally in terms of t, where $y = tx$ is the straight line joining the origin to this point. Any cross-ratio of the four points in which a straight line meets the quartic is then identical with the cross-ratio of the parameters of the point. This case is of no interest.

(ii) If two of the double-points of a unicursal quartic are confluent at the point $x = 0$, $y = 0$, and the third double-point is the point $x = 0$, $z = 0$, then the equation of the quartic may be written in the form

$$(yz - ax^2)^2 = x^2(bx - y)(x - cy),$$

where a, b, and c have any values which do not make the quartic degenerate. That is, we may not have $a = 0$, $b = 0$ simultaneously, $bc = 1$, or $a^2 = b$.*

If we put $$bx - y = t^2(x - cy),$$

we obtain the coordinates of any point on the curve rationally in terms of t in the form

$$x : y : z :: (t^2 - b)(ct^2 - 1) : (t^2 - b)^2 : (ct^2 - 1)\{a(ct^2 - 1) + (bc - 1)t\}.$$

The intersections of the quartic with the straight line $lx + my + nz = 0$ are given by the equation

$$(t^2 - b)(ct^2 - 1)l + (t^2 - b)^2 m + a(ct^2 - 1)^2 n + (bc - 1)(ct^3 - t)n = 0,$$

which written at length is

$$(cl + m + ac^2 n)t^4 + (bc - 1)cnt^3 - \{(bc + 1)l + 2bm + 2acn\}t^2 - (bc - 1)nt + bl + b^2 m + an = 0,$$

or $$Pt^4 - cRt^3 - (bP + Q + acR)t^2 + Rt + bQ + aR = 0,$$

where $P = cl + m + ac^2 n$, $Q = l + bm + acn$, $R = (1 - bc)n$.

For any proper quartic curve and an arbitrary straight line P, Q, and R are independent, and no one of them vanishes.

* I have to thank one of the referees for pointing out that I discussed this case originally with an equation that was not completely general.

Any cross-ratio of the four points is that of the four elements t_1^2, t_2^2, t_3^2, t_4^2, where t_1, t_2, t_3, t_4 are the parameters of the four points. Thus a cross-ratio of the four points and the corresponding cross-ratio of the four parameters are $(0, \xi_1, \xi_2, \xi_3)$ and $(\infty, \xi_1, \xi_2, \xi_3)$, where ξ_1 denotes $(t_1+t_4)(t_2+t_3)$, with similar meanings for ξ_2 and ξ_3. If these two cross-ratios are expressible in terms of each other for every straight line, a homogeneous relation must exist between ξ_1, ξ_2, and ξ_3, when P, Q, and R have any arbitrary values, no one of them being zero.

For a biquadratic $t^4+pt^3+qt^2+rt+s=0$,

the equation giving ξ is found by writing the biquadratic as

$$[t^2+\tfrac{1}{2}pt+\tfrac{1}{2}(q-\xi)]^2=(Mt+N)^2,$$

and is $$\xi^3-2q\xi^2+(q^2+pr-4s)\xi+p^2s-pqr+r^2=0;$$

which in the case under consideration becomes

$$P^3\xi^3-2P^2(bP+Q+acR)\xi^2+P[(bP+Q+acR)^2-cR^2-4P(bQ+aR)]\xi$$
$$+(bc-1)R^2(cQ-P)=0.$$

A *homogeneous* relation between ξ_1, ξ_2, ξ_3 is equivalent to the existence of a relation between

$$bP+Q+acR, \quad cR^2+4P(bQ+aR), \quad R^2(cQ-P),$$

or to the vanishing of a coefficient in the equation for ξ. The latter condition cannot be satisfied for an arbitrary straight line. Also P, Q, R are independent and different from zero, so that a relation of the kind required can only exist if the Jacobian of the three expressions above with respect to P, Q, and R vanishes.

The Jacobian is the product of R and

$$4b^2P^2+4bcQ^2+c(bc+1-2a^2c)R^2+(4ac-2abc^2)QR$$
$$-2aRP-4b(bc+1)PQ,$$

and consequently cannot vanish for all values of P, Q, and R in the case of any proper quartic curve.

Thus no unicursal quartic of this type possesses this property.

(iii) If a quartic has three distinct nodes, its equation may be obtained by writing $1/x$, $1/y$, $1/z$ for x, y, z in the equation of the conic

$$S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

where no one of a, b, c may vanish.

The intersections of the quartic with the straight line $px+qy+rz=0$ are then the inverses of the intersections of $S=0$ with the conic

$$S' \equiv 2pyz+2qzx+2rxy=0.$$

Any cross-ratio of the points in which the straight line meets the quartic is that which the four intersections of the conics subtend at a point of $S'=0$. The corresponding cross-ratio of the parameters is that which the four intersections subtend at a point of $S=0$. These may be taken to be respectively $(0, k_1, k_2, k_3)$ and (∞, k_1, k_2, k_3), where k_1, k_2, k_3 are the three values of k for which $kS+S'=0$ is a line-pair. If one of these is a function of the other, k_1, k_2, and k_3 must be connected by a homogeneous relation.

The equation which gives k is, in the ordinary notation of conics,

$$\Delta k^3+2(Fp+Gq+Hr)k^2+(ap^2+bq^2+cr^2-2fqr-2grp-2hpq)k$$
$$+2pqr=0.$$

When neither a, b, nor c vanishes, a homogeneous relation between k_1, k_2, k_3 can exist only when $F=0$, $G=0$, $H=0$, or

$$ap^2+bq^2+cr^2-2fqr-2grp-2hpq$$

is a multiple of $(Fp+Gq+Hr)^2$.

In the first case each nodal tangent is inflexional. In the second case each double point is a cusp.

In the first case the relation between k_1, k_2, k_3 is $k_1+k_2+k_3=0$. If μ is a cross-ratio of the four points in which a straight line meets the quartic, and μ' is the corresponding cross-ratio of the parameters, then

$$\mu' = (k_1-k_2)/(k_1-k_3) = (2k_2+k_3)/(k_2+2k_3),$$

so that $$k_3/k_2 = (\mu'-2)/(1-2\mu');$$

and hence $$\mu = k_3(k_1-k_2)/k_2(k_1-k_3) = \mu'(\mu'-2)/(1-2\mu').$$

In the second case the relation between k_1, k_2, and k_3 is

$$k_1^{\frac{1}{2}}+k_2^{\frac{1}{2}}+k_3^{\frac{1}{2}}=0.$$

Hence $$\mu' = k_3^{\frac{1}{2}}(k_3^{\frac{1}{2}}+2k_2^{\frac{1}{2}})/k_2^{\frac{1}{2}}(k_2^{\frac{1}{2}}+2k_3^{\frac{1}{2}}),$$

so that $$k_3^{\frac{1}{2}}/k_2^{\frac{1}{2}} = \mu'-1 \pm (\mu'^2-\mu'+1)^{\frac{1}{2}},$$

and $$\mu = \mu'\{\mu'-1 \pm (\mu'^2-\mu'+1)^{\frac{1}{2}}\}^2.$$

This relation in a rational form is

$$\mu'^4-4\mu'^3\mu+6\mu'^2\mu-4\mu'\mu+\mu^2=0.$$

By reciprocation we see that this is the relation connecting μ, a cross-ratio of the pencil formed by the four tangents to a nodal cubic from any point, with μ', the corresponding cross-ratio of the pencil joining the points of contact to the nodes.

From the last theorem we find by inversion that the same relation connects a cross-ratio of the pencil formed by the four tangents, drawn to a trinodal quartic from any point on it and distinct from the tangent at the point, with the corresponding cross-ratio of the parameters of the points of contact.

One of the referees has pointed out that the theorem with regard to the nodal cubic is the same as the property that the four tangent-planes from an arbitrary line to a twisted cubic have a cross-ratio expressible in terms of the cross-ratio of their points of contact, regarded as points on the curve. It is not difficult to see that this is true. For we can express the curve by $x/\theta^3=y/\theta^2=z/\theta=w/1$, in such a way that three arbitrary points on the curve are given by $\theta=0,\ \infty,\ 1$. There are two transversals of the tangents at these three points and at an arbitrary fourth point, so that the expression is two-valued.

3. I will now take the case of the nodal cubic in some detail.

(i) For projective purposes we may take the equation to be

$$x^3+y^3=3xyz,$$

whether the nodal tangents are real or imaginary. Here $z=0$ is the line of the inflexions, and $x=0,\ y=0$ are the nodal tangents.

The coordinates may be represented rationally in terms of a parameter t by the equations

$$x:y:z::3t:3t^2:1+t^3.$$

The parameters of three collinear points are connected by the relation

$$t_1t_2t_3+1=0,$$

and those of six points on a conic by the relation

$$t_1t_2t_3t_4t_5t_6=1.$$

The parameters of corresponding points, that is, points such that the tangents at them meet on the cubic, are of the form $\pm t$. The straight lines joining them to the node are harmonic conjugates with regard to the

nodal tangents. The equation of the straight line joining the points, the tangents at which meet on the cubic at the point t, is

$$x+t^2y+3tz=0,$$

which passes through the point $-t$, and also through the harmonic conjugate of the point t with respect to the node and the point in which the straight line joining the point to the node meets the line of inflexions. It is then easily constructed, whether the nodal tangents are real or not. The Hessian of the cubic is

$$x^3+y^3+xyz=0.$$

If the node and six points are given, we can construct by ruler and compasses the nodal tangents, the line of inflexions, the tangent at any point, the third point of intersection of the cubic with the straight line joining two points on it, the tangents from any point on the cubic to the cubic, distinct from the tangent at the point, the intersections of the cubic with any straight line drawn through a point on it, the point in which any straight line through the node meets the cubic again, points on the Hessian, and the polar conic of any point. I think it probable that such constructions have already been given somewhere, and therefore omit details of them here.

In what follows I shall frequently refer to the node as the point O.

(ii) The parameters of the points of contact of the tangents drawn to the cubic from the point X, Y, Z are the roots of the equation

$$Xt^4-2Yt^3+3Zt^2-2Xt+Y=0.$$

These four points lie on a conic with any pair of points whose parameters are connected by the relation

$$Yt_1t_2=X.$$

If we take the special pair of points whose parameters are given by the equation

$$Yt^2+X=0,$$

it can be shown without difficulty that the equation of the conic through the six points is

$$S' \equiv (Y^2+ZX)\,x^2+(X^2+YZ)\,y^2+3XYz^2 \\ -2Y^2yz-2X^2zx-2XYxy=0.$$

Another conic through the four points of contact is the polar conic of

X, Y, Z, namely

$$S \equiv Xx^2+Yy^2-Xyz-Yzx-Zxy = 0.$$

The general equation of a conic through the four points is then

$$kS+S' = 0.$$

The particular conic of this system which passes through X, Y, Z is easily found to be

$$ZS+S' = 0.$$

The equation which gives the three values of k, for which $kS+S' = 0$ is a line-pair, becomes, after removing the factor X^3+Y^3+XYZ,

$$k^3+3Zk^2+4(X^3+Y^3-3XYZ) = 0.$$

If μ, μ', λ are respectively three corresponding cross-ratios of the pencils joining the four points of contact to a point on $ZS+S' = 0$, the node, and a point on $S' = 0$, then

$$\mu = \{Z, k_1, k_2, k_3\}, \quad \mu' = \{\infty, k_1, k_2, k_3\},$$

$$\lambda = \{0, k_1, k_2, k_3\}.$$

Since $$k_1^{-1}+k_2^{-1}+k_3^{-1} = 0,$$

and also $$3Z = -k_1-k_2-k_3,$$

we find that

$$\mu = \lambda^3(\lambda-2)/(1-2\lambda), \quad \mu' = \lambda(\lambda-2)/(1-2\lambda),$$

and we can verify that μ and μ' are connected by the relation found before in § 2.

(iii) Strictly this analysis would not be applicable if X, Y, Z were on the Hessian, in which case $kS+S' = 0$ would be a line-pair for all values of k. In this case $S = 0$ breaks up into the factors

$$Xy+Yx = 0 \quad \text{and} \quad Xx/Y+Yy/X-z = 0,$$

and $S' = 0$ breaks up into

$$Xx/Y+Yy/X-z = 0 \quad \text{and} \quad -X^2x-Y^2y-3XYz = 0.$$

The values of k for which

$$k(Xy+Yx)-X^2x-Y^2y-3XYz = 0$$

passes through an intersection of the curve with the straight line

$$X^2x+Y^2y-XYz = 0$$

will be found to be given by the same cubic as before, so that the same relations hold between the corresponding cross-ratios.

(iv) Denote for the moment the cubic equation

$$k^3+3Zk+4(X^3+Y^3-3XYZ)=0$$

by $F(k)=0$; denote also X^3+Y^3-3XYZ by C.

If $F(k)=0$ is regarded as a biquadratic with an infinite root, its invariants I and J are respectively $3Z^2/4$ and $-(Z^3+2C)/8$, so that $27\mu'^2(\mu'-1)^2/4(\mu'^2-\mu'+1)^3$, which is $(I^3-27J^2)/I^3$, is equal to

$$-4C(C+Z^3)/Z^6.$$

The invariants of the biquadratic $(k-Z)F(k)=0$ are

$$3(Z^4-8CZ)/4 \quad \text{and} \quad (Z^6+20CZ^3-8C^2)/8,$$

so that $27\mu^2(\mu-1)^2/4(\mu^2-\mu+1)^3=-64C(C+Z^3)^3/Z^3(Z^3-8C)^3$.

By considering the invariants of the biquadratic $kF(k)=0$, we find similarly that

$$27\lambda^2(\lambda-1)^2/4(\lambda^2-\lambda+1)^3=(C+Z^3)/Z^3.$$

It follows that, when λ is given, the locus of X, Y, Z is a cubic of the system $C+pZ^3=0$. When μ is given, the locus is four cubics of this system; when μ' is given, the locus is two cubics of the system. This explains the degrees in λ, μ, and μ' of the algebraic equations which connect them.

If X, Y, Z is a point on the line of inflexions, $\mu'=-\omega$ or $-\omega^2$, and the corresponding values of μ' are $-\omega^2$ and $-\omega$, so that the pencil of tangents drawn from a point on the line of inflexions is equi-anharmonic, and so is the pencil which the points of contact subtend at the node.

The line of inflexions is not the complete locus of points such that the pencil of tangents is equi-anharmonic. The cubic $Z^3=8C$ is also part of the locus.

4. *The conic* $S'=0$.

(i) If we denote the point X, Y, Z by P then, as P describes a straight line through the node (the point O), the conic $S'=0$ passes through four fixed points, which lie on the two straight lines $Xx^2+Yy^2=0$. $S'=0$ is a line-pair also when P is on the cubic or on the Hessian. Denote these two positions of P by P_1 and P_2 respectively.

The line-pair corresponding to P_1 is the pair of tangents drawn to the cubic from P_1, distinct from the tangent at P_1.

The line-pair corresponding to P_2 consists of the straight line joining the points of contact of these tangents, and the straight line joining the points of contact of the tangents drawn to the Hessian from P_2, distinct from the tangent at P_2.

Since all these four points lie on the straight lines $Xx^2+Yy^2=0$, they are the four points through which $S'=0$ passes. The particular conic $S'=0$ corresponding to P is determined from the fact that the polar of P with respect to it is the line of inflexions, as may be easily verified.

(ii) If the nodal tangents are imaginary, or if, when they are real, X and Y have opposite signs, these four points are real, and $S'=0$ is real and easily constructed by ruler and compasses, if we suppose the cubic to be determined by its node and six other points. If X and Y have the same sign and the nodal tangents are real, the four points are imaginary.

The harmonic triangle of the quadrangle formed by the four points has its vertices at O and P and the point in which $X^2x+Y^2y=0$ meets the line of inflexions. The equations of its sides are

$$u \equiv X^2x+Y^2y-3XYz=0, \quad v \equiv X^2x+Y^2y=0, \quad w \equiv Xy-Yx=0.$$

The equation of $S'=0$ may be written in the form

$$u^2+(a-1)v^2+XY(a+3)w^2=0,$$

where a denotes Z/Z_1, X, Y, Z_1 being the point P_1.

If X and Y are of the same sign, the conic given by this equation is real or not according as a is less than or greater than unity, which is the condition that OP should meet the conic in real or imaginary points. So that, when the nodal tangents are real, $S'=0$, if real, can be constructed by ruler and compasses. For we can construct the sides of the self-conjugate triangle found above, and we shall have more than enough additional conditions to complete the construction.

(iii) Since the polar conic of P can be constructed by ruler and compasses, we have reduced the problem of drawing the tangents from any point to a nodal cubic, determined by its node and six other points, to the problem of finding the intersections of two conics which can be constructed by ruler and compasses; one of them is always real, the other is real if any of the tangents are real.

Such a construction is not very practical but is of theoretical interest. The only really practical construction would consist of tracing the curve and drawing the tangents by means of the eye.

(iv) An alternative construction depending on the intersections of conics, which are always real, and can be constructed by ruler and compasses, is the following.

The conic $z^2 = 4xy$ is the envelope of a straight line joining corresponding points on the Hessian, or of the straight line joining the two points, distinct from O, in which the nodal tangents meet the polar conic of any point on the cubic. This conic is therefore always real and easily constructed. If we take the coordinates of any point on it to be proportional to t^2, 1, $2t$, its intersections with the polar conic of X, Y, $-3Z$ are given by the equation

$$Xt^4 - 2Yt^3 + 3Zt^2 - 2Xt + Y = 0,$$

the equation giving the parameters of the points of contact of the tangents from P.

The line joining O to one of these intersections is $x - t_1^2 y = 0$. The line $x + t_1^2 y = 0$ is the harmonic conjugate of $x - t_1^2 y = 0$ with respect to the nodal tangents. Let $x + t_1^2 y = 0$ meet the cubic at Q. Q lies on the tangent at the point t, so that PQ is a tangent drawn from P.

The point X, Y, $-3Z$ can be found from X, Y, Z by a ruler construction.

5. (i) The polars of O with respect to all conics through the points of contact of the tangents from P pass through the intersection of the straight lines $X^2x + Y^2y + 3XYz = 0$ and $Yx + Xy = 0$. The locus of this point is the cubic $x^3 + y^3 + 3xyz = 0$. The point, which is the same for all positions of P on a straight line through O, is easily constructed.

For, if P_1' is the point on the cubic corresponding to P_1 [P_1 having the same meaning as in 4 (i)], the point required is the harmonic conjugate of P_1' with respect to O and the point in which OP_1' meets the line of inflexions.

(ii) We can now solve by a ruler and compasses construction the following problem :—Given two points on the cubic, to find the other two points, such that the tangents at the four points are concurrent.

Let A_1 and A_2 be the two points, and let the tangents at A_1 and A_2 meet at P. Let A_3 and A_4 be the other two points. Let A_1A_2 meet the cubic again at L, and let A_3A_4 meet the cubic again at M. Let LM meet the cubic again at N. Then ON and OP are harmonic conjugates with respect to the nodal tangents. For, if t_1, t_2, t_3, t_4 are the parameters of A_1, A_2, A_3, A_4, $Xt_1t_2t_3t_4 = Y$. Also the parameters of L and M are $-(t_1t_2)^{-1}$, $-(t_3t_4)^{-1}$ respectively, so that the parameter of N is $-t_1t_2t_3t_4$.

Hence N can be found, when OP is known ; and when N and L are

known, M can be found. Also, when N is known, the point of concurrency of the polars of O with respect to conics through A_1, A_2, A_3, A_4 is known. The straight line joining this point to O is divided harmonically by the straight lines A_1A_2 and A_3A_4. Hence we know another point on A_3A_4 in addition to M, and therefore A_3A_4 is known.

If A_1 and A_2 are real, P can be found by drawing the tangents at these points. If A_1 and A_2 are not real but given as the intersections of the cubic with a straight line drawn through some real point L on the cubic, then P may be found as follows. (Of course this method can be applied if A_1 and A_2 are real. The actual position of P is not needed for the construction of A_3A_4, only the line OP, but it is more satisfactory to be able to construct P in all cases.)

The pairs of further points, in which a straight line through L, a point of the cubic, meet the cubic again, subtend an involution at O, and subtend another involution at L', the point corresponding to L. These facts may be easily verified.

By considering two consecutive positions of the straight line we verify the following facts. If Q is the harmonic conjugate of L with respect to A_1A_2, which is the point in which A_1A_2 meets the polar conic of L, then OP and OQ belong to the first involution, and $L'P$ and $L'Q$ belong to the second involution.

Moreover the tangent at Q to the polar conic of L passes through P. We thus have three easily constructed straight lines, each of which passes through P. The first of these straight lines is sufficient for the determination of A_3A_4, for which the actual position of P is unnecessary.

ON THE CONVERGENCE OF THE DERIVED SERIES OF FOURIER SERIES

By PROF. W. H. YOUNG.

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LEMMA 1.—*If $f(x)$ is absolutely integrable in (a, b), then*

$$\int_x^y f(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx \to 0,$$

uniformly, as $n \to \infty$, and $a < x < y < b$.

LEMMA 2.—*If in addition $g(x)$ have bounded variation, then*

$$\int_a^b f(x+u)\, g(u) \sin nu\, du \to 0,$$

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(i) *converges uniformly* ($C1$),

(ii) *converges boundedly* ($C1$),

(iii) *oscillates uniformly above, or below* ($C1$),

(iv) *oscillates boundedly* ($C1$),

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The former series converges (Cp), if the latter converges ($C, p-1$).

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(B) $[f(x+t)-f(x-t)-2Kt]/t^2$ is absolutely integrable in the neighbourhood of $t = 0$.

(C) $f(x)$ has a generalised differential coefficient at the point x, and

$$\frac{1}{t}\int_0^t |\, d\,[f(x+t)-f(x-t)]\,|$$

is a bounded function of t in the neighbourhood of $t = 0$.

(D) $$\frac{1}{t}\int_0^t \{[f(x+t)+f(x-t)]/t\}\, dt$$

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The former series converges (Cp) if the latter converges ($C, p-2$).

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1. The derived series of Fourier series may themselves be Fourier series. They then require no special consideration. In the contrary case the series of which they are the derived series may be associated with a particular type of function, *e.g.* with a function of bounded variation; or the function of which we have the Fourier series may not be known to have any particular properties, except in the neighbourhood of a particular point.

I have already had occasion to point out that the derived series of Fourier series of functions of bounded variation possess some of the properties of Fourier series. I propose to illustrate the interest of these series further.

But besides these series, which possess such properties in the whole interval of periodicity, there are trigonometrical series, scarcely less important, which possess the properties in question, or closely analogous ones, if not in the whole interval, at least in a portion or portions of it. These series are got by differentiating the Fourier series of functions which in such portion or portions of the interval of periodicity are more

or less highly specialised, while they may be perfectly general in the rest of the interval.

It is easy to give examples. The derived series even of such Fourier series as $\Sigma n^{-1} \cos nx$ and $\Sigma n^{-1} \sin nx$, do not converge ordinarily, but do so when summed by the method of Cesàro, index unity, throughout the whole of the completely open interval of periodicity from which the origin is excluded, and converge in the manner in question precisely to the differential coefficients of the functions with which these Fourier series are associated.

We notice further that, if we continue to differentiate these Fourier series we obtain trigonometrical series which, in the completely open interval in question, always converge to the correspondingly higher differential coefficient provided only the index of the Cesàro summation employed increases by unity at each successive differentiation.

These are comparatively trivial illustrations of a general theory. Two main cases present themselves, according as we may, with propriety, regard the derived series considered as associated, in the sub-interval or intervals to which it is restricted to converge, with a function, or not. If the Fourier series from which our series is got by derivation p times is associated with a function which in the interval in question is a p-th integral, and accordingly possesses almost everywhere in that interval a p-th differential coefficient, this p-th differential coefficient may with propriety be regarded as associated with the p-th derived series. We might then conveniently call such a p-th derived series a *restricted Fourier series*. *Mutatis mutandis* we are then able to enunciate for restricted Fourier series almost all the theorems which hold good for Fourier series; the chief difference is that we must employ Cesàro convergence index p, if the process of derivation has been employed p times, and that we must be careful to restrict their application to the sub-interval or intervals for which the p-th differential coefficient exists.

But besides these derived series there are derived series which cannot be said in the same sense to have a function associated with them. We already had a simple example of such a series. It is the series whose general term is $\cos nx$, and is accordingly the derived series of the Fourier series of a function of bounded variation, whose differential coefficient is everywhere negative unity, except at the origin, where it is $+\infty$; but it is not convenient to regard this differential coefficient as associated with the series. More generally there may be no p-th differential coefficient, and yet the p-th derived series may, in some sub-interval, possess many of the properties of Fourier series.

It is convenient then to consider separately these two classes of derived series, and I propose to do so in the present communication.

In the course of the argument, which is based on reasoning analogous to that I have already employed in a previous communication, I require certain theorems with regard to the convergence of the Fourier series, and I take the opportunity of giving new tests for the convergence of a Fourier series.* These tests enable me at the same time to give a more general form to the tests for the Cesàro convergence of a Fourier series, also contained in the previous communication to this Society just referred to.

The main object of the paper is, as the title states, the convergence of the derived series of a Fourier series. This includes also the discussion of its uniform convergence, and occasion is taken to prove for ordinary Fourier series one or two elementary theorems in a somewhat simpler manner than usual. As an illustration of the use of uniform convergence, certain theorems which I have given in earlier communications are then extended to restricted Fourier series.

One of the fundamental results obtained is the property that the convergence of a Fourier series at a point, and the Cesàro convergence,† index p, of its p-th derived series at a point, depend merely on the form of the function in the neighbourhood of the point, and that this property only just holds in general. In other words, if we replace the convergence of the Fourier series by Cesàro convergence $k-1$, where $k<1$, or the Cesàro convergence (Cp), by Cesàro convergence $(C, p-1+k)$, the independence in question ceases in general to exist. I have thought it worth while in this connection to prove one or two additional theorems. In the case of the Fourier series, if the function has bounded variation elsewhere than at the point at least, and in the case of the p-th derived series, if a corresponding restriction holds, the index of convergence may be reduced.

It appears, however, that the primary question for investigation with respect to the convergence of the p-th derived series of a Fourier series is its convergence in the Cesàro manner index p. In fact it is for this index and no lower one that the convergence at the point x depends only on the nature of the function in the neighbourhood of the point considered. Furthermore, it is for this index, and no lower one, that tests for convergence of the usual type exist. We may almost say that the concept of

* The most important case of these tests is that in which the function of bounded variation which occurs in them is an integral. The test, in the form which it then takes, has been already published in a communication I have recently made to the Académie des Sciences.

† See the second footnote to § 7, below.

Cesàro convergence, index p, is naturally associated with trigonometrical series which are the p-th derived series of Fourier series.

Again, when such a series does not converge (Cp), it may still, in certain cases, be employed just as if it did do so.

The question naturally suggests itself whether, if instead of differentiating, we integrate, there are analogous results. I have thought it relevant to the matter in hand to give the simple and easily obtained theorems in this connection. It will be seen that, if we integrate a Fourier series p times, the trigonometrical series thus obtained necessarily converges in the Cesàro manner, index $-p$ (minus p), everywhere.

What happens if, instead of starting with proper Fourier series, and differentiating and integrating them, we start with the derived series of Fourier series of functions of bounded variation? As already pointed out, these more general series behave in some respects *exactly* like Fourier series; from our present point of view, however, they behave *almost* like Fourier series, the difference being precisely analogous to that between a closed and an open interval. We find that the same theorems hold, provided only the index of Cesàro summation be increased by a positive quantity k, as small as we please. Here, as elsewhere in our theorems, we mean by $(C, p+k)$ not merely that the index is $p+k$, in the ordinary sense; it is only necessary to suppose that any kind of Cesàro convergence of positive type, logarithmic or otherwise, is superposed to the (Cp) convergence, for the theorems to hold. Indeed all that is necessary is that the superposed operation should be such that, when performed on S_n, the typical term of a sequence for which ΣS_n is bounded, the result is zero.

This result illustrates the fact that, if the p-th derived series (in particular if the first derived series), of the Fourier series of $f(x)$ is to converge at the point x with a lower index of convergence than the normal p (or in the particular case unity), the function $f(x)$ must have special properties elsewhere than in the neighbourhood of the point x; in the case of the theorems just considered the reduction of the index is *almost unity*.

The question naturally arises as to whether the reduction unity itself may not be secured by a suitable condition with regard to the behaviour of $f(x)$, and in particular the important problem suggests itself as to whether the first derived series of a Fourier series may not in certain cases converge ordinarily, without being a Fourier series. The answer to this question is certainly in the affirmative. We can indeed give three important classes of Fourier series for which such ordinary convergence is possible, though not customary. These are

(1) *The allied series of Fourier series, not themselves Fourier series.*

(2) *Pseudo-Fourier series, that is trigonometrical series whose coefficients can be expressed in the Fourier form as absolutely convergent integrals, without the associated function possessing an absolutely convergent integral.*

(3) *More generally trigonometrical series whose coefficients can be expressed in the Fourier form as non-absolutely convergent integrals, such as Harnack-Lebesgue integrals, or what I have called Y-integrals.*

As to the first of these classes, I have on various occasions given tests for the ordinary convergence of an allied series. These tests merely involve assumptions with respect to the nature of the associated function of the Fourier series in a sub-interval of the interval of periodicity, and by no means presuppose this function to have such properties in the whole interval as to secure that the allied series is also a Fourier series, whatever restrictions we impose with regard to a particular sub-interval.

As an instance of the second class, a pseudo-Fourier series which at the same time comes under the head of the first class, we have

$$\Sigma (\sin nx)/\log n,$$

which is the derived series, as well as the allied series, of a Fourier series, and converges everywhere without exception although not itself a Fourier series.*

Conditions of space and time have prevented me on the present occasion from discussing the derived series of the allied series of a Fourier series, but it should be noted that these are derived series of Fourier series. In fact the allied series itself of a Fourier series, though not in general a Fourier series, is as remarked above always the first derived series of a Fourier series, namely that of the function

$$\frac{1}{\pi}\int_0^\pi [F(x+u)-F(x-u)]\cot \tfrac{1}{2}u\,du,$$

where $F(x)$ is an integral of the function $f(x)$, associated with the Fourier series, to which our series is allied. Consequently the allied series of the Fourier series of a function $f(x)$ may be treated by the methods of this paper.

We thus see, for example, that this allied series converges $(C1)$ at the

* Its coefficients b_n are expressible in the Fourier form, the coefficients a_n are all zero and are only expressible in the Fourier form if Cauchy's principal values be utilised.

point x if the function

$$g(u) = \int_0^\pi \frac{1}{uv}[F(x+u+v)-F(x-u+v)-F(x+u-v)+F(x-u-v)]\,dv$$

has bounded variation.

Again, the allied series of the Fourier series of a function which in a sub-interval of the interval of periodicity has its square summable, is a restricted Fourier series of the first class, whose associated function has its square summable in its region of existence. Or yet again, the allied series of a restricted Fourier series of Class 1 is a restricted Fourier series of a similar type, but of Class 2.

It may be remarked in the same connection not only that the allied series of the Fourier series of an integral is itself a Fourier series, from which our result that the allied series of a Fourier series is the first derived series of a Fourier series was deduced, but that the same is true of the allied series of the Fourier series of a function of bounded variation, so that the allied series of the first derived series of the Fourier series of a function of bounded variation is itself the first derived series of the Fourier series of a function of bounded variation.

Moreover, the allied series of the Fourier series of a function of bounded variation converges almost everywhere, not merely in the ordinary way, but even in the Cesàro manner with any negative fractional index $k > -1$. On the other hand, the allied series of the Fourier series of an integral converges almost everywhere $(C, -1)$.

However, besides these theorems so obtained we have others precisely analogous to the theorems of the present paper. They are related to them in the same way as the tests for the convergence of the allied series are to the corresponding tests for that of the Fourier series. I have, however, as already stated, thought it advisable not to enter into these matters on the present occasion.

2. We shall first consider various conditions for the convergence of a Fourier series. (§§ 2–9.)

We shall require the following formula :—

$$\tfrac{1}{2}\pi C(na_n) \equiv (a_1+2a_2+\ldots+na_n)/n$$

$$= \frac{1}{2n}\int_0^\pi \frac{d}{dx}\,[\cot \tfrac{1}{2}x(1-\cos nx)]f(x)\,dx - \tfrac{1}{2}a_n, \quad (1)$$

$f(x)$ being the even function associated with the Fourier series $\Sigma a_n \cos nx$.

This is easily proved as follows :—

$$\begin{aligned}
\tfrac{1}{2}\pi C(na_n) &= C\left(\int_0^\pi n\cos nx\right) f(x)\,dx \\
&= \int_0^\pi C(n\cos nx)\, f(x)\,dx \\
&= \int_0^\pi C\left(\frac{d}{dx}\sin nx\right) f(x)\,dx \\
&= \int_0^\pi \frac{d}{dx} C(\sin nx)\, f(x)\,dx \\
&= \frac{1}{2n}\int_0^\pi \frac{d}{dx}\left[\cot\tfrac{1}{2}x - \operatorname{cosec}\tfrac{1}{2}x\cos(n+\tfrac{1}{2})x\right] f(x)\,dx \\
&= \frac{1}{2n}\int_0^\pi \frac{d}{dx}(\cot\tfrac{1}{2}x - \cot\tfrac{1}{2}x\cos nx - \sin nx)\, f(x)\,dx,
\end{aligned}$$

which becomes at once the right-hand side of (1).

Since by the Theorem of Riemann-Lebesgue a_n has the unique limit zero, it follows from (1) that, in considering the limits of $C(na_n)$, it is only the integral on the right-hand side of the formula (1) which need be retained.

Hence it immediately follows that *the limits of* $C(na_n)$ *are independent of the form of the function* $f(x)$ *except in the immediate neighbourhood of the origin*, since, e being any small positive quantity,

$$(1-\cos nx)\frac{d}{dx}\cot\tfrac{1}{2}x \quad \text{and} \quad \cot\tfrac{1}{2}x$$

lie between fixed finite bounds when x lies in the interval (e, π) and n increases indefinitely, so that our integral from e to π is the sum of two integrals, of which the first

$$\frac{1}{2n}\int_e^\pi (1-\cos nx)\, f(x)\frac{d}{dx}\cot\tfrac{1}{2}x\,dx$$

vanishes in virtue of the factor $1/2n$, and the second

$$\int_e^\pi \tfrac{1}{2}\cot\tfrac{1}{2}x\, f(x)\sin nx\,dx$$

vanishes by the theorem of Riemann-Lebesgue.

3. In considering the limits of $C(na_n)$, we may further change $\cot\frac{1}{2}x$

into $1/\frac{1}{2}x$ in the formula (1), which accordingly becomes

$$\operatorname*{Lt}_{n\to\infty} \tfrac{1}{2}\pi C(na_n) = \operatorname*{Lt}_{n\to\infty} \frac{1}{n}\int_0^e \frac{d}{dx}\left(\frac{1-\cos nx}{x}\right) f(x)\,dx, \qquad (2)$$

where e is as small a positive quantity as we please, independent of n.

Indeed since

$$\frac{1}{n}\frac{d}{dx}\left(\cot\tfrac{1}{2}x - \frac{1}{\frac{1}{2}x}\right)(1-\cos nx)$$

$$= \frac{1}{n}(1-\cos nx)\frac{d}{dx}\left(\cot\tfrac{1}{2}x - \frac{1}{\frac{1}{2}x}\right) + \sin nx\left(\cot\tfrac{1}{2}x - \frac{1}{\frac{1}{2}x}\right),$$

which is the sum of two terms, the first of which is the product of $1/n$ into a bounded function of x and n, and the second the product of $\sin nx$ into a bounded function of x, for values of x in the closed interval $(0, e)$, we see that the difference of the two integrals in (1) and (2) is the sum of two integrals, of which the first vanishes uniformly in virtue of the factor $1/n$ in the integrand, and the second by the theorem of Riemann-Lebesgue, when n increases indefinitely. This proves the formula (2).

4. We can now prove the following theorem:—

THEOREM.—*If $\Sigma a_n \cos nx$ is the Fourier series of an even function $f(x)$, then*

$$\operatorname*{Lt}_{n\to\infty} C(na_n) = 0,$$

provided

(i) *$f(x)$ is simply discontinuous at the origin, and*

(ii) $\dfrac{1}{x}\displaystyle\int_0^x |\,d[xf(x)]\,|$ *is a bounded function of x in a certain neighbourhood of the origin.*

In fact, in virtue of the condition (i), we may change the lower limit of integration in the formula (2) from zero to $p = 2P\pi/n$, where P is as large an integer as we please, n being correspondingly larger. To prove this let us write t/n for x; $f(t/n)$ will then be bounded and have $f(+0)$ for limit as n increases indefinitely. We may therefore multiply by the bounded function $\dfrac{d}{dx}\left(\dfrac{1-\cos t}{t}\right)$, and integrate term-by-term. This gives

$$\operatorname*{Lt}_{n\to\infty} \frac{1}{n}\int_0^p \frac{d}{dx}\left(\frac{1-\cos nx}{x}\right) f(x)\,dx = \operatorname*{Lt}_{n\to\infty} \int_0^{2P\pi} f(t/n)\frac{d}{dt}\left(\frac{1-\cos t}{t}\right) dt$$

$$= \int_0^{2P\pi} f(+0)\frac{d}{dt}\left(\frac{1-\cos t}{t}\right) dt = 0.$$

We thus have

$$\operatorname*{Lt}_{n\to\infty} \tfrac{1}{2}\pi C(na_n) = \operatorname*{Lt}_{n\to\infty} \int_p^e f(x)\,\frac{d}{dx}\left(\frac{1-\cos nx}{nx}\right)dx = \operatorname*{Lt}_{n\to\infty} \int_p^e xf(x)\frac{d}{dx}\,q_n(x)\,dx,$$

where $$q_n(x) = \frac{1-\cos nx}{nx^2} - \frac{1}{n}\int_x^e \frac{1-\cos nx}{x^3}\,dx.$$

Now, by the condition (ii), $xf(x)$ must be a function of bounded variation; for $\int |\,d[xf(x)]\,|$ is the total variation of $xf(x)$, and is, by the condition (ii), bounded.

We may therefore integrate by parts, and write

$$\operatorname*{Lt}_{n\to\infty} \tfrac{1}{2}\pi C(na_n) = \operatorname*{Lt}_{n\to\infty} \Big[xf(x)\,q_n(x)\Big]_p^e - \int_p^e q_n(x)\,d[xf(x)].$$

Now $$|\,q_n(x)\,| \leqslant \frac{2}{nx^2} + \frac{1}{n}\int_x^e \frac{2}{x^3}\,dx = \frac{3}{nx^2}.$$

Therefore

$$\left|\operatorname*{Lt}_{n\to\infty} \tfrac{1}{6}\pi C(na_n)\right| \leqslant \operatorname*{Lt}_{n\to\infty} \frac{1}{n}\left\{\left|\left[\frac{f(x)}{x}\right]_p^e\right| + \int_p^e \frac{|\,d[xf(x)]\,|}{x^2}\right.$$

$$= f(p)/2P\pi + \operatorname*{Lt}_{n\to\infty} \frac{1}{n}\left(\int_p^e \frac{d\phi(x)}{x^2}\right),$$

where $$\phi(x) = \int_0^x |\,d[xf(x)]\,|.$$

But, integrating by parts,

$$\frac{1}{n}\int_p^e \frac{d\phi(x)}{x^2} = \frac{1}{n}\left[\frac{\phi(x)}{x^2}\right]_p^e + \frac{2}{n}\int_p^e \frac{\phi(x)}{x^3}\,dx$$

$$\leqslant \phi(e)/ne^2 + B/2P\pi + \frac{2B}{n}\int_p^e \frac{dx}{x^2},$$

where B denotes the upper bound of $\dfrac{1}{x}\displaystyle\int_0^x |\,d[xf(x)]\,|$, which, by the condition (ii), is finite. Letting n increase indefinitely, the right-hand side of the last relation approaches $3B/2P\pi$. Hence

$$\left|\operatorname*{Lt}_{n\to\infty} \tfrac{1}{6}\pi C(na_n)\right| \leqslant \frac{f(p)+3B}{2P\pi}.$$

Since, by (1), $f(p)$ remains bounded as n increases indefinitely, and P is as large as we please, the right-hand side of the last relation is as small as we please, which proves the theorem.

5. Hence we have the following theorem on the convergence of a Fourier series.

THEOREM. — *If, as h approaches zero, $\frac{1}{2}[f(x+h)+f(x-h)]$ has a unique limit C, and $\frac{1}{h}\int^h |d[h\{f(x+h)+f(x-h)\}]|$ is a bounded function of h in a certain neighbourhood surrounding the point $h=0$, however small this neighbourhood may be, then the Fourier series of $f(x)$ converges to C at the point x.**

Since

$$\tfrac{1}{2}[f(x+u)+f(x-u)] \sim \Sigma\,(a_r \cos rx + b_r \sin rx)\cos ru,$$

we only have to replace the function $f(x)$ by the even function of u, $\frac{1}{2}[f(x+u)+f(x-u)]$, to reduce the problem to that of an even function having a unique limit at the origin, and to replace the point x by the origin. We shall therefore only need to prove the theorem for the case $x=0$, f being an even function.

Now writing

$$s_n = a_1+a_2+\ldots+a_n, \qquad C(na_n) = (a_1+2a_2+\ldots+na_n)/n,$$

we have
$$s_n - C(na_n) = (s_1+s_2+\ldots+s_{n-1})/n. \tag{3}$$

Now, by the preceding theorem (§ 4), $C(na_n)$ has the unique limit zero, when n increases indefinitely. Also, since $f(x)$ is bounded in the neighbourhood of the origin, by hypothesis, and approaches a unique limit $f(+0)$, the Fourier series converges when summed in the Cesàro manner, index unity, so that the right-hand side of (3) also approaches a unique limit. Thus, by (3), s_n approaches the same unique limit, that is, the Fourier series converges at the origin. This proves the theorem.

COR. 1.—*If $\frac{1}{2}[f(x+h)+f(x-h)] \to C$, as $h \to 0$, and in a certain neighbourhood of $h=0$ we have*

$$\tfrac{1}{2}[f(x+h)+f(x-h)] = \frac{1}{h}\int^h g(t)\,dt,$$

where $\frac{1}{h}\int_0^h |g(t)|\,dt$ is a bounded function of h, the Fourier series of $f(x)$ converges to C at the point x.†

* The enunciation of this theorem was made by me to the Société Helvétique des Sciences Naturelles, in August of the present year, at Schuls in the Engadine.

† This was stated and proved by me in a recent note in the *Comptes rendus*.

COR. 2.—*If* $\frac{1}{2}[f(x+u)+f(x-u)] \to C$, *as* $u \to 0$, *and, in a certain neighbourhood of* $u = 0$, *the function of* u

$$\tfrac{1}{2}u[f(x+u)+f(x-u)]$$

has bounded derivates, then the Fourier series of $f(x)$ *converges to* C *at the point* x.

For in this case the conditions of Cor. 1 are satisfied, $g(t)$ being any one of the derivates of $\frac{1}{2}u[f(x+u)+f(x-u)]$. This corollary is, in fact, identical with the first corollary in the case in which $g(t)$ is a bounded function of t.

The first condition imposed shows that the second condition of this corollary is equivalent to the hypothesis that *the derivates of*

$$\tfrac{1}{2}[f(x+u)+f(x-u)],$$

with respect to u, *when multiplied by* u, *should be bounded functions of* u *in a certain neighbourhood of* $u = 0$.

Using the more general theorem that a function whose derivates are absolutely integrable (summable), and finite except at a countable set of points, is the integral of its derivates, we have the following also:—

COR. 3.—*If* $\frac{1}{2}[f(x+u)+f(x-u)] \to C$, *as* $u \to 0$, *and, in a certain neighbourhood of* $u = 0$, *the derivates of the function of* u

$$\tfrac{1}{2}[f(x+u)+f(x-u)]$$

are, when multiplied by u, *absolutely integrable, and, except possibly at a countable set of points, finite, then the Fourier series of* $f(x)$ *converges to* C *at the point* x.

6. Remembering that the Fourier series of $f(x)$ converges (Cp) at the point x, if that of

$$\phi_1(u) = \frac{1}{u}\int_0^u \tfrac{1}{2}[f(x+t)+f(x-t)\,dt = \int_0^1 \tfrac{1}{2}[f(x+ut)+f(x-ut)]\,dt$$

converges $(C, p-1)$, we get the following additional corollaries:—

COR. 4.—*The Fourier series of* $f(x)$ *converges* $(C1)$ *at the point* x *if*

$\phi_1(+0)$ *exists* and*

$$\frac{1}{u}\int_0^u |\, d[t\phi_1(t)]\,| \equiv \frac{1}{u}\int_0^u \tfrac{1}{2}\,|f(x+t)+f(x-t)|\,dt$$

$$= \int_0^1 \tfrac{1}{2}\,|f(x+ut)+f(x-ut)|\,dt$$

is a bounded function of u in some interval containing $u=0$. The Cesàro sum of the series is then $\phi_1(+0)$.

COR. 5.—*The Fourier series of $f(x)$ converges (C2) at the point x, if $\phi_2(+0)$ exists, where*

$$\phi_2(u) \equiv \frac{1}{u}\int_0^u \phi_1(t)\,dt = \int_0^1\int_0^1 \tfrac{1}{2}[f(x+ut_1t_2)+f(x-ut_1t_2)]\,dt_1\,dt_2,$$

and

$$\frac{1}{u}\int_0^u |\, d[t\phi_2(t)]\,| \doteq \frac{1}{u}\int_0^u |\,\phi_1(t)\,dt\,|$$

$$= \int_0^1 \left|\int_0^1 \tfrac{1}{2}\,[f(x+ut_1t_2)+f(x-ut_1t_2)]\,dt_1\right|\,dt_2$$

is a bounded function of u in some interval containing $u=0$. The second Cesàro sum of the series is then $\phi_2(+0)$.

COR. 6.—*The Fourier series of $f(x)$ converges (Cp) at the point x, if $\phi_p(+0)$ exists, where*

$$\phi_p(u) = \frac{1}{u}\int_0^u \phi_{p-1}(t)\,dt$$

$$= \int_0^1\int_0^1\cdots\int_0^1 \tfrac{1}{2}\,[f(x+ut_1t_2\ldots t_p)+f(x-ut_1t_2\ldots t_p)]\,dt_1\,dt_2\ldots dt_p,$$

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* The condition that $\phi_1(+0)$ should exist is, of course, the same as that

$$\int_0^u \tfrac{1}{2}\,[f(x+t)+f(x-t)]\,dt$$

should have a differential coefficient at $u=0$, and the same remark applies to $\phi_p(+0)$ and $\int_0^u \phi_{p-1}(t)\,dt$. These conditions given above may be compared with the less general conditions of similar form given on p. 267 of *The Convergence of a Fourier Series and its Allied Series*, in the second line of which "index p" should read "index $(p+1)$."

and if $\frac{1}{u}\int_0^u |\phi_{p-1}(u)du|$, *that is*

$$\int_0^1 \left| \int_0^1 \cdots \int_0^1 | \tfrac{1}{2}[f(x+ut_1t_2 \ldots t_p)+f(x-ut_1t_2 \ldots t_p)]\, dt_1 dt_2 \ldots dt_{p-1}| \, dt_p,$$

is a bounded function of u *in some interval containing* $u=0$. *The* p*-th Cesàro sum of the Fourier series of* $f(x)$ *is then* $\phi_p(+0)$.

7. With regard to the preceding theorem (§ 5), it should be noticed that a considerable extension is possible if we use the notion of *Cesàro summation with negative index.*

Denoting the ordinary partial summation of a Fourier series by s_n and the Cesàro partial summation, index minus one, by t_n, we have by a definition which naturally suggests itself

$$ns_n = t_1+t_2+\ldots+t_n, \tag{1}$$

and therefore $t_n = ns_n-(n-1)\,s_{n-1} = s_{n-1}+n\,(s_n-s_{n-1})$

$$= s_{n-1}+n\,(a_n \cos nx+b_n \sin nx)\,; \tag{2}$$

where $$f(x) \sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx),$$

and therefore

$$\tfrac{1}{2}[f(x+u)+f(x-u)] \sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)\cos nu.$$

We see therefore, from (2), that t_n and s_n have the same limit at the point x, or the same limits, provided the coefficients of the derived series of the Fourier series of the even function

$$\tfrac{1}{2}[f(x+u)+f(x-u)]$$

converge to zero.

In particular, using the theorem of Riemann-Lebesgue, and the generalisation of it which I have recently given,* we have the following

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* This theorem was given in its original form by Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe", 1854, *Ges. Werke*, p. 254, and states that, if $f(x)$ has a Riemann integral, the coefficients of its Fourier series a_n and b_n converge to zero, as $n \to \infty$. The extension to any summable function was given by Lebesgue, 1903, *Annales sc. de l'école normale sup.*, Sér. 3, Vol. xx, quoted in Hobson's *Theory of Functions of a Real Variable*, p. 674.

In other words, *the theorem of Riemann-Lebesgue states that the coefficients of the derived*

results :—

The Fourier series of an integral converges $(C, -1)$ *everywhere to that integral.*

The Fourier series of a function of bounded variation converges everywhere $(C, k-1)$,* *where* k *is any positive quantity whatever.*

8. We can then assert that (1) *the question of the convergence* $(C, k-1)$, *where* $0 < k$, *depends only on the nature of the function in the neighbourhood of the point considered, provided the function be known to be a function of bounded variation outside this interval;* also (2) the argument we have used, involving as it does such quantities as $1/n$ to the first power only, and therefore still applicable when the index is k, where $0 < k$, shows that, so far as the neighbourhood $(0, e)$ is concerned, $C(na_n)$, that is $(C1)(na_n)$, may be replaced by $(Ck)(na_n)$, where $0 < k \leqslant 1$. Thus we have the theorem that *the conditions of the theorem of* § 5, *or of any of*

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series of the Fourier series of an integral converge to zero. The generalisation of this theorem which I have given recently in a paper "On the Order of Magnitude of the Coefficients of a Fourier Series", presented to the Royal Society, is as follows :—*The coefficients of the derived series of the Fourier series of an even function of bounded variation converge to zero, when the convergence is taken in the Cesàro manner, index unity, and those of an odd function of bounded variation converge in the same manner to* $1/\pi$ *times the jump of the function at the origin.*

* The definition of Cesàro convergence, index $k-1$, here adopted, may, in the first instance, be taken to be that explained in my paper, already cited, from these *Proceedings*, Vol. 10, p. 264, the Cesàro summations with negative integral indices being first defined. The reader will have no difficulty with these latter; he merely has to repeat the process involved in equation (1) above. The definition so obtained is necessarily equivalent, for values of the index between 0 and -1, to that devised by Knopp and Chapman for such values, in this sense, that, if a series converges $(C, k-1)$ in their sense for all such values of the index, it will converge $(C, k-1)$ in my sense, and conversely.

With regard to Cesàro convergence of positive index, I assume tacitly here as elsewhere the known equivalence of the various definitions hitherto proposed for such convergence, and employ them indifferently, as the circumstances of the demonstration may render convenient.

Two remarks should be made about the theorem in the text :—In the first place, this theorem is an immediate consequence of the classical result that the coefficients of the derived series of the Fourier series of a function of bounded variation are bounded, provided we make use of a general theorem in the arithmetic theory of series (which is, however, more difficult than would be otherwise necessary), namely one given by Hardy and Littlewood in these *Proceedings*, Ser. 2, Vol. 11, p. 462, Theorem 37. In the second place it should be remarked that we may assert the truth of the theorem when the expression "converges $(C, k-1)$" is interpreted in the still larger sense in which Cesàro convergence of any positive type is superimposed, in the manner explained in the footnote to my paper already quoted, on the Cesàro convergence, index -1.

its corollaries, ensure not only the convergence of the Fourier series, but its convergence $(C, k-1)$, *provided only* $f(x)$ *is in addition a function of bounded variation in the part of the interval of periodicity outside the neighbourhood of the origin mentioned in the conditions.*

We have thus two important cases in which the Fourier series converges $(C, k-1)$, where $0 < k$.

(1) *When* $f(x)$ *has bounded variation in the whole interval of periodicity.*

(2) *When* (i) $f(x)$ *has bounded variation in every interval not containing the point at which the convergence is considered, while at the point itself* $f(x)$ *has a unique limit, or at least* $f(x+u)+f(x-u)$ *has a unique limit, as* $u \to 0$, *and in addition* (ii)

$$\frac{1}{u}\int^{u} \left| d\{u[f(x+u)+f(x-u)]\} \right|$$

is a bounded function of u *in the neighbourhood of* $u = 0$.

This is a remarkable extension for $(C, k-1)$ of Dirichlet's theorem, properly generalised, and bears the same relation to this generalisation that de la Vallée Poussin's condition does to Dirichlet's original test.

If we require the condition (ii) to hold throughout the interval of periodicity, we have corresponding results. For, in an interval not containing $u = 0$, the condition (ii) only has a meaning when $f(x)$ has bounded variation; and conversely, if $f(x)$ has bounded variation, our condition fulfils itself.

9. The following test for the convergence of the Fourier series of $f(x)$ is slightly more general than that commonly employed.* It requires that

$$\frac{f(x+2u)-f(x-2u)-K}{u}$$

should be absolutely integrable in an interval containing the origin, K *being a suitable constant.*

In fact, denoting by s_n the n-th partial summation of the Fourier series of $f(x)$, we have, by the usual formula,

$$s_n = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} [f(x+2u)+f(x-2u)] \operatorname{cosec} u \sin(2n+1)\, u\, du.$$

* Hobson's *Theory of Functions of a Real Variable*, pp. 680, 681; where, however, it is assumed that $f(x+2u)+f(x-2u)$ has, as $u \to 0$, a unique limit which is the constant K.

Also $$\operatorname*{Lt}_{n\to\infty} \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} K \operatorname{cosec} u \sin(2n+1)u\,du = \tfrac{1}{2}K.$$

Now in consequence of our hypothesis,

$$\frac{f(x+2u)+f(x-2u)-K}{u}\,\frac{u}{\sin u}$$

is absolutely integrable in some interval containing the origin, and is therefore absolutely integrable from 0 to $\frac{1}{2}\pi$. Therefore, by the theorem of Riemann-Lebesgue, when we multiply by $\sin(2n+1)u$ and integrate from 0 to $\frac{1}{2}\pi$, we get a value which vanishes as $n\to\infty$. That is, by the above,

$$\operatorname*{Lt}_{n\to\infty} s_n - \tfrac{1}{2}K = 0,$$

which proves that the Fourier series of $f(x)$ converges to $\frac{1}{2}K$.

10. It is well known that the convergence, or mode of oscillation, or more precisely the nature of the upper and lower functions, of a Fourier series at a point depend only on the nature of the function in a portion of the interval of periodicity as small as we please containing the point. Corresponding to this fact in the ordinary theory, we have the following theorem.

THEOREM.—*The upper and lower functions of the p-th derived series of a Fourier series at a particular point depend only on the nature of the function associated with the Fourier series in a neighbourhood enclosing the point, as small as we please, provided the summation of the series is performed in the Cesàro manner, index p.*

We will first prove this theorem for the first derived series.

Writing

$$u_n(t) = \tfrac{1}{2}+\cos t+\ldots+\cos(n-1)t = \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\sin(n-\tfrac{1}{2})t,$$

and $$C(u_n) = (u_1+u_2+\ldots+u_n)/n = (1-\cos nt)/2n(1-\cos t),$$

we have evidently $$\frac{d}{dt}C(u_n) = C\left(\frac{d}{dt}(u_n)\right),$$

that is,

$$\frac{d}{dt}\left[(1-\cos nt)/2n(1-\cos t)\right] = C\left[-\sin t-2\sin 2t-\ldots-(n-1)\sin(n-1)t\right]. \tag{1}$$

Now $$\tfrac{1}{2}\left[f(x+t)-f(x-t)\right] \sim \Sigma(b_r\cos rx - a_r\sin rx)\sin rt. \tag{2}$$

We may therefore multiply both sides of the relation (2) by the continuous function $\frac{d}{dt}C(u_n)$, and integrate term-by-term. Thus

$$\tfrac{1}{2}\int_{-\pi}^{\pi}[f(x+t)-f(x-t)]\,\frac{d}{dt}[(1-\cos nt)/2n(1-\cos t)]\,dt$$

$$=\Sigma\int_{-\pi}^{\pi}(b_r\cos rx-a_r\sin rx)\sin rt\,[C(-\sin t-2\sin 2t-\ldots$$

$$-(n-1)\sin(n-1)\,t]\,dt.$$

Now $\int_{-\pi}^{\pi}\sin rt\sin st=0$, unless $r=s$, in which case it is equal to π. Therefore the right-hand side of the last equation becomes

$$-\pi C\{(b_1\cos x-a_1\sin x)+2(b_2\cos 2x-a_2\sin 2x)+\ldots$$

$$+(n-1)[b_{n-1}\cos(n+1)\,x-a_{n-1}\sin(n-1)x]\}.$$

Thus, denoting by $r_n(x)$ the n-th Cesàro partial summation of the derived series of the Fourier series of $f(x)$, we have

$$r_n(x)=-\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x+t)-f(x-t)]\,\frac{d}{dt}[(1-\cos nt)/2n(1-\cos t)]\,dt. \quad (3)$$

The integrand in this last integral is an even function of t, so that the integral is twice the same integral from 0 to π. Hence, in order to prove the theorem enunciated in § 9, it is only necessary to prove that the same integral from e to π, where e is any small positive quantity, vanishes when n increases indefinitely.

$$\text{Now}\quad \frac{d}{dt}[(1-\cos nt)/(1-\cos t)]=\frac{n\sin nt}{1-\cos t}-\frac{(1-\cos nt)\sin t}{(1-\cos t)^2}. \quad (4)$$

Thus our integral is the difference of the two integrals obtained by multiplying the two terms on the right-hand side of (4) by $[f(x+t)-f(x-t)]/n$ and integrating from e to π. The first of these two integrals is zero by the theorem of Riemann-Lebesgue, since

$$[f(x+t)-f(x-t)]/(1-\cos t)$$

is absolutely integrable in the interval considered. The second term on the right of (4) is numerically less than $2/(1-\cos e)^2$, so that the second integral vanishes in virtue of the factor $1/n$.

This proves the theorem for the first derived series.

11. When p is greater than unity, we proceed as follows to prove the theorem of § 10.

Using the generalisations of the sine and cosine discussed by me in an earlier paper,* so that

$$C_p(t) = \frac{t^p}{\Gamma(p+1)}\left[1 - t^2/(p+1)(p+2) + t^4/(p+1)(p+2)(p+3)(p+4) - \ldots\right], \tag{1}$$

we have,† denoting by $(Cp)[S_n(x)]$ the Cesàro n-th partial summation of index p of our series,

$$\begin{aligned}(Cp)[S_n(x)] &= \tfrac{1}{2}a_0 + \sum_{m=1}^{n} A_m(1-m/n)^p \\ &= \frac{\Gamma(p+1)}{\pi}\int_0^\infty t^{-p-1}\, C_{p+1}(t)\,[f(x+t/n)+f(x-t/n)]\,dt \\ &= \Gamma(p+1)\, I/\pi, \end{aligned} \tag{2}$$

where, integrating by parts,

$$\begin{aligned} I = n\Big[t^{-p-1}C_{p+1}(t)[F(x+t/n)-F(x-t/n)]\Big]_0^\infty \\ -n\int_0^\infty \frac{d}{dt}\left[t^{-p-1}C_{p+1}(t)\right][F(x+t/n)-F(x-t/n)]\,dt. \end{aligned} \tag{3}$$

Now $t^{-p-1}C_{p+1}$ vanishes when t is infinite, and $[F(x+t/n)-F(x-t/n)]$ is zero when t is zero. Therefore the square bracket expression disappears, and we are left with the integral on the right alone.

Differentiating with respect to x, we may differentiate under the sign of integration, as will be seen by the following reasoning. Let us write $G(x)$ for $\int_0^x F(x)\,dx$. Then, since $F(x)$ is continuous, it is the differential co-

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* "On Infinite Integrals involving a Generalisation of the Sine and Cosine Functions," 1912, *Quarterly Journal*, Vol. 43, pp. 161–177

† *Loc. cit.*, p. 177.

efficient of $G(x)$. Hence, as $h \to 0$,

$$\frac{1}{h}\int_0^t [F(x+h+t/n)-F(x+t/n)]\,dt$$

$$= n\int_x^{x+t/n} \frac{1}{h}[F(\xi+h)-F(\xi)]\,d\xi$$

$$= n\,[G(x+h+t/n)-G(x+t/n)-G(x+h)+G(x)]/h$$

$$\to n\,[F(x+t/n)-F(x)]$$

$$= n\int_x^{x+t/n} f(\xi)\,d\xi = \int_0^t f(x+t/n)\,dt,$$

and the convergence is bounded, since the bounds of the incrementary ratio of $G(x)$ are the same as those of $F(x)$.

By a known theorem,* therefore, if $g(t)$ is any function of bounded variation in the infinite interval, with zero as its unique limit when $t \to \infty$, we may introduce the factor $g(t)$ on each side under the sign of integration, and replace the upper limit of integration by infinity. Thus we have

$$\operatorname*{Lt}_{h\to 0} \frac{1}{h}\int_0^\infty g(t)\,[F(x+h+t/n)-F(x+t/n)]\,dt = \int_0^\infty g(t)\,f(x+t/n)\,dt,$$

that is

$$\frac{d}{dx}\int_0^\infty g(t)\,[F(x+t/n)]\,dt = \int_0^\infty g(t)\,f(x+t/n)\,dt,$$

and hence also

$$\frac{d}{dx}\int_0^\infty g(t)\,[F(x+t/n)-F(x-t/n)]\,dt = \int_0^\infty g(t)\,[f(x+t/n)-f(x-t/n)]\,dt.$$

It remains therefore only to show that if we put

$$g(t) = \frac{d}{dt}\,[t^{-p-1}C_{p+1}(t)] = -(p+1)t^{-p-2}C_{p+1}(t)+t^{-p-1}C_p(t),$$

$g(t)$ vanishes at infinity, and is a function of bounded variation in the whole infinite interval, to justify the statement that we may differentiate under the sign of integration in (1).

That $g(t)$ vanishes at infinity is at once evident from its expression in terms of C_p and C_{p+1}; that it is a function of bounded variation in the

* W. H. Young, "The Application of Expansions to Definite Integrals", 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, § 10, pp. 475–477.

whole infinite interval may be, for instance, proved by differentiating it, which gives

$$g'(t) = (p+1)(p+2)\,t^{-p-3}\,C_{p+1}(t) - 2(p+1)\,t^{-p-2}\,C_p(t) + t^{-p-1}\,C_{p-1}(t),$$

and shows at once that $g'(t)$ is summable in the infinite interval, whence $g(t)$ is a function of bounded variation in the whole infinite interval, provided that $p \geqslant 1$.

We have therefore, from (3),

$$\frac{dI}{dx} = -\int_0^\infty n[f(x+t/n) - f(x-t/n)]\,\frac{d}{dt}(t^{-p-1}C_{p+1})\,dt \quad (p \geqslant 1), \quad (4)$$

which is of the same form as I, $n\dfrac{d}{dt}(t^{-p-1}C_{p+1})$ taking the place of $t^{-p-1}C_{p+1}$ as the multiplier of $f(x+t/n)$ and $f(x-t/n)$ in the integrand.

We may therefore repeat the process, provided p is sufficiently large; for one differentiation we had to have $p \geqslant 1$, for two we must have $p \geqslant 2$, and so on. Thus, finally,

$$\frac{d^r I}{dx^r} = (-)^r \int_0^\infty n^r [f(x+t/n) + (-)^r f(x-t/n)]\,\frac{d^r}{dt^r}[t^{-p-1}C_{p+1}(t)]\,dt \quad (r \leqslant p). \quad (5)$$

Using Leibnitz's theorem for the expansion of

$$\frac{d^r}{dt^r}[t^{-p-1}C_{p+1}(t)],$$

we get, since, apart from a numerical factor,

$$\frac{d^r}{dt^r}\,t^{-p-1}\left(\frac{d^{p-r}}{dt^{p-r}}\,C_{p+1}\right) = t^{-p-r-1}C_{r+1},$$

$$\frac{d^p I}{dx^p} = \int_0^\infty n^r [f(x+t/n) + (-)^p f(x-t/n)] \sum_{r=0}^{p} A_r\,t^{-p-r-1}C_{r+1}(t)\,dt, \quad (6)$$

or, say,

$$\frac{d^p I}{dx^p} = \sum_{r=0}^{p} A_r \int_0^\infty K_r(t)\,dt, \quad (6')$$

where, since $C_{r+1} = -C_{r-1} + t^{r-1}/(r-1)!$,

$$K_r(t) = -t^{-2}K_{r-2}(t) + n^p[f(x+t/n) + (-)^p f(x-t/n)]\,t^{-p-2}/(r-1)!. \quad (7)$$

Now what we have to show, in order to prove our theorem, is that

$\frac{d^p I}{dx^p}$ is independent of the form of $f(u)$, except for a small interval of values of u enclosing the point $u = x$. That is, we have to show that, if we change the lower limit of integration in (6), or (6′), from zero to ne, the integral vanishes in the limit, when $n \to \infty$; it will therefore be sufficient to show that this is the case with the individual integrals $\int_{ne}^{\infty} K_r(t)\,dt$.

Now, by (7), this will be the case, provided it is true for $\int_{ne}^{\infty} t^{-2} K_{r-2}(t)\,dt$, and for

$$\int_{ne}^{\infty} n^p [f(x+t/n)+(-)^p f(x-t/n)]\, t^{-p-2}\,dt$$

$$= \frac{1}{n}\int_{e}^{\infty} [f(x+u)+(-)^p f(x-u)]\, u^{-p-2}\,du. \quad (8)$$

The latter integral vanishes in virtue of the factor $1/n$, and the former integral will do so also, provided $\int_{ne}^{\infty} K_{r-2}(t)\,dt$ does so. Thus, by induction, the theorem is true, provided $\int_{ne}^{\infty} K_2(t)\,dt$ and $\int_{ne}^{\infty} K_1(t)\,dt$ vanish in the limit when $n \to \infty$.

Now $\int_{ne}^{\infty} K_1(t)\,dt = \int_{ne}^{\infty} n^p [f(x+t/n)+(-)^p f(x-t/n)]\, t^{-p-1} \sin t\,dt$

$$= \int_{e}^{\infty} [f(x+u)+(-)^p f(x-u)]\, u^{-p-1} \sin nu\,du,$$

which vanishes by the theorem of Riemann-Lebesgue; also

$$\int_{ne}^{\infty} K_2(t)\,dt = \int_{ne}^{\infty} n^p [f(x+t/n)+(-)^p f(x-t/n)]\, t^{-p-2}(1-\cos t)\,dt$$

$$= \frac{1}{n}\int_{e}^{\infty} [f(x+u)+(-)^p f(x-u)]\, u^{-p-2}(1-\cos nu)\,du,$$

which vanishes when $n \to \infty$, since it is the sum of the integral in (8) and an integral which vanishes by the theorem of Riemann-Lebesgue.

This therefore proves the theorem.

12. An immediate consequence of the preceding theorem is the following generalisation of a theorem I have given previously.*

THEOREM.—*If $f(x)$ be throughout a certain interval (a, b) a function of bounded variation, then the derived series of the Fourier series of $f(x)$ converges $(C1)$, almost everywhere in the interval in question, to the differential coefficient of the function.*

In fact the convergence $(C1)$ in the interval in question does not depend on the nature of the function outside that interval, and is therefore the same as if $f(x)$ had everywhere bounded variation. But in this latter case the series in question converges (Ck), where $0 < k$, and therefore certainly converges $(C1)$ almost everywhere. This proves the theorem.

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* W. H. Young, "The Usual Convergence of a Class of Trigonometrical Series", 1913, *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 21–23. The reader will have noticed that by a clerical error (Ck) has been omitted in the enunciation of the theorem, which should be as follows:—"The derived series of the Fourier series of a function of bounded variation converges (Ck), $0 < k$, almost everywhere to the differential coefficient of the function." In the course of the proof another clerical error has crept in, which I take this opportunity of correcting. In line 17, p. 22, *delete*

$$\text{const. } n^{-k} \int_1^\infty |\, df(x+t)\,| \,;$$

line 18 should then read:—"this last integral existing, since $f(x)$ is a periodic function of bounded variation throughout the whole infinite interval. Thus this term is of the order n^{-k}."

The argument is here perhaps unduly condensed. That the fact quoted does ensure the existence of the integral is, for instance, easily seen if we integrate by parts from 1 to B. We then have

$$\int_1^B t^{-1-k} \,|\, df(x+t)\,| = \int_1^B t^{-1-k}\, dF(t) = \Big[(1+t)^{-1-k} F(t)\Big]_1^B + (1+k)\int_1^B t^{-2-k} F(t)\, dt,$$

where, since f is periodic, we have, writing

$$1 + 2r\pi \leqslant t < 1 + 2(r+1)\pi,$$

$$F(t) = \int_1^t |\, df(x+t)\,| \leqslant \int_1^{1+2(r+1)\pi} |\, df(x+t)\,| = (r+1)\, K \leqslant \frac{K}{2\pi}(t - 1 + 2\pi),$$

where

$$K = \int_1^{1+2\pi} |\, df(x+t)\,| \,.$$

Hence

$$\Big[(1+t)^{-1-k} F(t)\Big]_1^B \to 0,$$

as $B \to \infty$, and $\int_1^\infty t^{-2-k} F(t)\, dt$ exists, which proves the existence of the integral under consideration.

13. Again we immediately have the following theorem, which is a particular case of a more general theorem to be proved later (§ 22, Theorem C):—

THEOREM.—*If $f(x)$ has a derivate $f_1(x)$ which is continuous at the point x,* then the derived series of the Fourier series of $f(x)$ converges $(C1)$ at the point x to $f_1(x)$.*

In fact, since, as we have seen, the convergence of the series in question is independent of the form of $f(x)$ except in the neighbourhood of the point x considered, we may assume that $f(x)$ is an integral except in an interval containing the point x in which $f_1(x)$ is bounded; this latter hypothesis is allowable, since $f_1(x)$ is continuous at the point x.

Since a function with a bounded derivate is an integral, $f(x)$ is now an integral in the whole closed interval of periodicity, and therefore the derived series of the Fourier series of $f(x)$ is a Fourier series, and its associated function is any one of the derivates of $f(x)$, and may therefore be taken to be $f_1(x)$. Since $f_1(x)$ is continuous at the point x, the Fourier series of $f_1(x)$ converges $(C1)$ to $f_1(x)$ at the point x. That is, the derived series of the Fourier series of the modified function $f(x)$ converges $(C1)$ to $f_1(x)$ at the point x.

Since the modification of $f(x)$ did not affect the convergence of the derived series, this proves the theorem.

COR.—*It is sufficient if $\frac{1}{2}[f(x+u)-f(x-u)]$ has a derivate with respect to u which is continuous at $u=0$; the derived series of the Fourier series of $f(x)$ then converges $(C1)$ to the value of this derivate at $u=0$.*

For $\frac{1}{2}[f(x+u)-f(x-u)] \sim \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \sin nu,$

x being here a constant, and u the variable. By the above theorem the derived series (with respect to u) of the above Fourier series converges $(C1)$ at $u=0$ to the derivate in question. That is

$$\left(\frac{d}{du}\,\frac{1}{2}[f(x+u)-f(x-u)]\right)_{u=0} = \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx),$$

which proves the corollary.

---

* All the derivates then coincide at the point x, so that there is a differential coefficient and all of them are continuous.

14. We now turn to a theorem of a different character, from which we shall be able to draw a variety of important consequences.

THEOREM.—*If two functions agree in a certain completely open interval, the difference of the n-th partial summations (C1) of the derived series of their Fourier series converges uniformly to zero in any closed interval inside the completely open interval.*

To prove this we require two Lemmas:—

LEMMA 1.*—*If $f(x)$ is a summable function in an interval (a, b), then, for any pair of points x and y of this interval,*

$$\operatorname*{Lt}_{n\to\infty} \int_x^y f(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx = 0,$$

the convergence being uniform, as x and y vary in (a, b).

If $f(x)$ is a constant, say P,

$$\left| \int_x^y P \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx \right| = \left| \frac{1}{n} P \left[\begin{matrix}\sin\\ \cos\end{matrix} nx\right]_x^y \right| \leqslant 2P/n,$$

from which the result is obvious.

Hence also it is obvious if $f(x)$ is a simple l- or u-function. Next, let $f_1, f_2, \ldots$ be functions none of which exceed $\leqslant f(x)$, and whose integrals have for limit the integral of $f(x)$. This is, for instance, the case if the functions f_r form a monotone ascending sequence with $f(x)$ as limit, or if they are properly chosen u-functions less than $f(x)$. Then, since $f - f_r \geqslant 0$,

$$\left| \int_x^y (f-f_r) \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx \right| \leqslant \int_x^y (f-f_r)\,dx \leqslant \int_a^b (f-f_r)\,dx \leqslant e,$$

if r is chosen greater than a certain fixed integer r_e, depending only on e.

This integral converges therefore uniformly to zero, when x and y vary as we please in (a, b) and $n \to \infty$. But

$$\int_x^y f(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx = \int_x^y (f-f_r) \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx + \int_x^y f_r(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx,$$

and each of the integrals on the right converges uniformly to zero, so that the integral on the left does so. This proves the theorem for $f(x)$.

Similarly the theorem holds for $f(x)$, if

$$f_r(x) \geqslant f(x).$$

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* This includes the result to which I habitually refer as the theorem of Riemann-Lebesgue.

Hence the theorem is true for semi-continuous functions, since these are the limits of monotone sequences of simple l- and u-functions; and therefore it is true for any bounded function $f(x)$, since the integral of $f(x)$ is the upper (lower) bound of the integrals of upper (lower) semi-continuous functions less (greater) than $f(x)$. Hence it is true for any positive summable function $f(x)$, as the limit of a monotone sequence of bounded functions. Finally, it is true of any summable function, as the difference of two positive summable functions.

This proves the lemma.

LEMMA 2.—*If $f(x)$ be summable in (a, b), and $g(x)$ have bounded variation, then $\int_a^b f(x+u)\, g(u) \sin nu\, du$ approaches zero uniformly as $n \to \infty$.*

Put $F(u) = \int_0^u f(x+t) \sin nt\, dt$: then, integrating by parts,

$$\int_a^b f(x+u)\, g(u) \sin nu\, du = F(b)\, g(b) - F(a)\, g(a) - \int_a^b F(u)\, dg(u). \qquad (1)$$

Now $$F(u) = \cos nx \int_x^{x+u} f(t) \sin nt\, dt - \sin nx \int_x^{x+u} f(t) \cos nt\, dt,$$

and therefore approaches zero uniformly as $n \to \infty$, since $|\sin nx|$ and $|\cos nx|$ do not exceed 1, and the two integrals approach zero uniformly, by Lemma 1. Integrating therefore with respect to the function $g(u)$ of bounded variation, $\int_a^b F(u)\, dg(u)$ approaches zero uniformly. Since also $F(b)\, g(b)$ and $F(a)\, g(a)$ approach zero, by the theorem of Riemann-Lebesgue, the required result follows from (1).

15. We now return to the theorem enunciated above (§ 14), and proceed to prove it.

Let (a, b) be the completely open interval in which f_1 and f_2 agree, and let us consider any point x of the closed interval (a', b') inside (a, b), where the distances of the end-points of the latter from those of the former interval are all greater than e. Then in the interval $(x-e, x+e)$ the functions f_1 and f_2 coincide.

Let $S'_{n,1}$ and $S'_{n,2}$ denote the differential coefficients of the n-th partial summations of the corresponding Fourier series, each taken in the Cesàro manner, index unity. Then, by the formula (3) of § 10,

$$[S'_{n,1}(x) - S'_{n,2}(x)] = -\int_0^\pi [\phi(x+t) - \phi(x-t)] \frac{d}{dt}[(1-\cos nt)/2n(1-\cos t)]dt,$$

where $$\phi(x) = f_1(x) - f_2(x)$$

is zero in the given interval. Hence

$$[S'_{n,1}(x) - S'_{n,2}(x)] = \tfrac{1}{2}\int_e^\pi \frac{\phi(x+t)-\phi(x+t)}{1-\cos t} \sin nt\, dt$$

$$-\frac{1}{2n}\int_e^\pi \frac{\phi(x+t)-\phi(x-t)}{(1-\cos t)^2}(1-\cos nt)\sin t\, dt.$$

The first of these integrals is the sum of two integrals, one involving $\phi(x+t)$ and the other $\phi(x-t)$, each of which converges uniformly to zero, by Lemma 2, as x moves about in the interval (a', b') and $n \to \infty$, since the quantity e does not then depend upon x. The second of the integrals on the right of the last equation also approaches zero uniformly; for, in virtue of the inequality

$$\left| \int_e^\pi \frac{\phi(x+t)-\phi(x-t)}{(1-\cos t)^2}(1-\cos nt)\sin t\, dt \right|$$

$$\leqslant 2\int_e^\pi |\phi(x+t)-\phi(x-t)|\, dt \Big/ (1-\cos e)^2$$

$$\leqslant 2\left\{\int_{e+x}^{\pi+x} |\phi(t)|\, dt + \int_{x-\pi}^{x-e} |\phi(t)|\, dt\right\} \Big/ (1-\cos e)^2$$

$$\leqslant 4\int_{a-\pi}^{b+\pi} |\phi(t)|\, dt/(1-\cos e)^2,$$

the integral in question is numerically less than $K/2n$, where K is independent of both n and x, as long as x remains in (a', b').

Thus $S'_{n,1} - S'_{n,2}$ also converges uniformly to zero, when x moves about in the closed interval (a', b') and $n \to \infty$.

This proves the theorem.

16. From the theorem just proved we can deduce a number of important consequences. The following is an immediate corollary :—

COR. :—

(i) *If two functions agree in a certain completely open interval, then if one of the derived series of the corresponding Fourier series converges uniformly in a closed interval or at a point of that open interval, so does the other.*

(ii) *If one series converges boundedly, so does the other.*

(iii) *If one series oscillates uniformly above or below, so does the other.*

(iv) *If one series oscillates boundedly, so does the other.*

*It is supposed that in all four cases the convergence is taken C*1.

17. But further the theorem gives us numerous theorems which correspond to the theorems in my paper "On the Integration of Fourier Series."* The following three theorems are given as applications of the principles, and as showing the use we can make of the derived series of Fourier series. These theorems all refer to a certain sub-interval of the interval of periodicity, and show that, in such an interval, these series are as useful as Fourier series.

THEOREM.—*If $f(x)$ is a function which, in a certain sub-interval of the interval of periodicity, is the integral of a function whose $(1+p)$-th power is summable, and $g(x)$ of a function whose $(1+1/p)$-th power is summable, then, if we multiply the derived series of the Fourier series of $f(x)$ by $g(x)$, we may integrate term-by-term over the sub-interval, provided we sum in the Cesàro manner index unity; and the result will be the integral of $g(x)\,df/dx$ over the sub-interval.*

Let $f_1(x)$ agree with $f(x)$ in the sub-interval, say (a, b), and outside (a, b) let $f_1(x)$ be the integral of a function whose $(1+p)$-th power is summable. Then the theorem is true for $f_1(x)$ and $g(x)$.

But in (a, b) $$\frac{df}{dx} = \frac{df_1}{dx},$$

so that $$\int_a^b g(x)\frac{df}{dx}\,dx = \operatorname*{Lt}_{n\to\infty}\int_a^b g(x)\,v_{n,1}(x)\,dx,$$

writing $v_n(x)$ and $v_{n,1}(x)$ for the n-th Cesàro partial summations of the derived series of the Fourier series of $f(x)$ and $f_1(x)$ respectively. But, by the theorem of § 14, $[v_n(x)-v_{n,1}(x)]$ converges uniformly to zero, as $n\to\infty$, x moving as we please in (a, b). Also $g(x)$ is summable, so that

$$\int_a^b g(x)\,[v_n(x)-v_{n,1}(x)]\,dx \to 0$$

* *Proc. London Math. Soc.*, 1910, Ser. 2, Vol. 9, pp. 449–462.

as $n \to \infty$; whence

$$\int_a^b g(x) \frac{df}{dx} dx = \operatorname*{Lt}_{n\to\infty} \int_a^b g(x)\, v_n(x) dx,$$

which proves the theorem.

18. The argument is the same in the proof of each of the two following theorems :—

THEOREM.—*If $f(x)$ is an absolutely convergent integral in a certain interval (a, b), and $g(x)$ any bounded function, the same result holds as in the preceding theorem, viz.,*

$$\int_a^b g(x) \frac{df}{dx} dx = \operatorname*{Lt}_{n\to\infty} \int_a^b g(x)\, v_n(x) dx,$$

where $v_n(x)$ is the Cesàro partial summation, index unity, of the derived series of the Fourier series of $f(x)$.

And also the following :—

THEOREM.—*If $f(x)$ be the integral of a bounded function in a certain interval (a, b), and $g(x)$ any summable function, the same result holds as in the last two theorems, viz.,*

$$\int_a^b g(x) \frac{df}{dx} dx = \operatorname*{Lt}_{n\to\infty} \int_a^b g(x)\, v_n(x) dx.$$

It may be remarked that in the first of these theorems there is no simplification when $p = 1$, nor in the second when $g(x)$ is a function of bounded variation, since the theorem respecting the uniformity (§ 14), refers only to the Cesàro summations, and not to the ordinary summations.

19. For reasons which will immediately appear, it will be convenient to introduce the term "restricted Fourier series", using this expression in the following sense :—

The p-th derived series of the Fourier series of $f(x)$ is said to be a restricted Fourier series of the p-th class, and to be restricted to one or more intervals (a, b), if, throughout each such completely open interval (a, b), $f(x)$ is a p-th integral.

The interval (a, b) may be a sub-interval of the interval $(-\pi, \pi)$ of periodicity, but it may also coincide with it without the restricted Fourier

series becoming itself a Fourier series; in fact, in order that a trigonometrical series may be a Fourier series it is necessary and sufficient that it should be the derived series of a function of period 2π which throughout the closed interval of periodicity is an integral.

The p-th differential coefficient of $f(x)$ in the interval (a, b) is called the function associated with the restricted Fourier series.

Instead of saying that a series is a restricted Fourier series, restricted to the interval (a, b), we may, for brevity, say it is *a Fourier series restricted to the interval (a, b)*.

With this definition the theorem of § 16 leads to the following important result:—

THEOREM.—*The condition that a restricted Fourier series of the p-th class should converge (Cp), at the point x of the interval to which it is restricted, is precisely of the same form as the condition that a Fourier series should converge (Cp) at that point, the function associated with the restricted Fourier series taking the place of that associated with the Fourier series.*

On the other hand, it will be noticed that there are for such a series no theorems relating to convergence (Cq), where $q < p$.

To prove this theorem consider, for simplicity, the case when $p = 1$, and let $f(x)$ be the function associated with the given restricted Fourier series, and v_n the n-th Cesàro partial summation of that series. On the other hand, let $f_1(x)$ be a function of period 2π, equal to $f(x)$ in the completely open interval (a, b), and an integral at the remaining points of the interval of periodicity, and let $v_{n,1}$ denote the n-th Cesàro partial summation of the Fourier series of $f_1(x)$. Then, by the theorem of § 16, $v_n(x) - v_{n,1}(x)$ converges uniformly to zero. Therefore, if any condition is satisfied which ensures the convergence of $v_{n,1}(x)$ to a unique and finite limit, $v_n(x)$ will also converge to the same limit. This condition will, of course, involve the function $f_1(x)$, but, since it is independent of the form of $f_1(x)$ except in a certain neighbourhood of the point x as small as we please, it will be unaltered* if we change $f_1(x)$ into $f(x)$, the point x being a point of the completely open interval (a, b).

* In all the conditions which actually are used, an arbitrary small neighbourhood of the point x appears explicitly as the range of the variable. It is hardly necessary to point out that, if the condition should involve formally values of $f_1(x)$ outside such an arbitrary small neighbourhood of the point, $f(x)$ may, when substituted, be supposed to have any convenient values at points outside (a, b), or, if we prefer, the interval (a, b) may in the condition be substituted for $(-\pi, \pi)$.

The condition being therefore fulfilled when $f(x)$ is substituted for $f_1(x)$, $v_n(x)$ converges. This proves the theorem.

In particular we have the condition corresponding to de la Vallée Poussin's condition for the convergence of a Fourier series :—

COR. 1.—*The restricted Fourier series of $f(x)$ converges (Cp) at the point x of the interval to which it is restricted, if*

$$\phi_{p+1}(u) = \int_0^1 \int_0^1 \dots \int_0^1 \tfrac{1}{2}[f(x+ut_1t_2\dots t_{p+1})+f(x-ut_1t_2\dots t_{p+1})]\,dt_1dt_2\dots dt_{p+1}$$

is a function of u of bounded variation in some interval containing $u=0$. The Cesàro sum of the series, index p, is then $\phi_{p+1}(+0)$.

We have also the condition corresponding to my own condition for the convergence of a Fourier series, given above (§ 5) :—

COR. 2.—*The restricted Fourier series of $f(x)$ converges (Cp) at the point x of the interval to which it is restricted, if $\phi_p(+0)$ exists, where*

$$\phi_p(u) = \frac{1}{u}\int_0^u \phi_{p-1}(t)\,dt$$

$$= \int_0^1 \int_0^1 \dots \int_0^1 \tfrac{1}{2}[f(x+ut_1t_2\dots t_p)+f(x-ut_1t_2\dots t_p)]\,dt_1dt_2\dots dt_p,$$

and if

$$\frac{1}{u}\int_0^u |\,dt\phi_p(t)\,| = \int_0^1 \Big| \int_0^1 \dots \int_0^1 \tfrac{1}{2}[f(x+ut_1t_2\dots t_p) + f(x-ut_1t_2\dots t_p)]\,dt_1dt_2\dots \Big|\,dt_p,$$

is a bounded function of u in some interval containing $u=0$. The p-th Cesàro sum of the restricted Fourier series of $f(x)$ is then $\phi_p(+0)$.

20. Among the derived series of Fourier series, restricted Fourier series play an important part, owing largely to the existence of a definite function with which such a series may be associated. The theorem just given, interesting as it is, is only a special case of more general results. In order that the r-th derived series of a Fourier series should converge (Cr) at a point, or in a certain interval, it is by no means necessary that the series should be a restricted Fourier series. We proceed to give theorems of this more general nature, confining our attention in the first instance to the case in which $r=1$. Special attention will be called to the cases when, in consequence of the hypotheses made, we are in effect dealing with restricted Fourier series. Before doing so, however, one general remark should be made. Our theory reveals the importance of the

condition for the convergence (Cr)* of a Fourier series. Whereas a Fourier series may be treated in general without going beyond $r = 1$, a restricted Fourier series cannot.

21. We shall require the following formulæ, which I had already given in a previous communication† :—

If
$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx), \tag{1}$$

and
$$g(u) \sim \sum_{n=1} A_n \cos(2n-1)u, \tag{2}$$

where
$$g(u) = \tfrac{1}{4}\{f(x+2u) - f(x-2u)\} \operatorname{cosec} u, \tag{3}$$

$g(u)$ being supposed to be absolutely integrable (summable), we have

$$\begin{aligned} A_n - A_{n+1} &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\,[\cos(2n-1)u - \cos(2n+1)u]\,du \\ &= \frac{2}{\pi}\int_{-\pi}^{\pi} g(u) \sin u \sin 2nu\,du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}\{f(x+2u) - f(x-2u)\} \sin 2nu\,du \\ &= b_n \sin nx - a_n \sin nx. \end{aligned} \tag{4}$$

Now let us write s_n for the n-th partial summation‡ of the derived series of (1), so that

$$s_n = \sum_{r=1}^{n-1} (rb_r \cos rx - ra_r \sin rx),$$

and S_n for the sum of the first n quantities A_r, so that

$$S_n = \sum_{r=1}^{n} A_r.$$

We then have, multiplying both sides of (4) by n, and summing,

$$s_n = S_n - nA_n. \tag{I}$$

Thus, since

$$C(s_n) = (s_1 + s_2 + \ldots + s_n)/(n+1),$$

---

* Sufficient conditions are, of course, obtained by replacing r by any smaller value, including zero.

† W. H. Young, "The Convergence of a Fourier Series and its Allied Series", 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 262, 263, § 4. The notation is altered here, $f(x)$, which is not an integral, taking the place of the integral $F(x)$.

‡ The term corresponding to $r = 0$ is, of course, zero.

and $$C(S_n) = (S_1+S_2+\ldots+S_n)/(n+1) = S_n - C(nA_n),$$

we have $$C(s_n) = 2C(S_n) - S_n. \qquad \text{(II)}$$

Hence, repeating the process of taking the mean any number p of times,

$$(Cp)(s_n) = 2(Cp)(S_n) - (C, p-1)S_n. \qquad \text{(III)}$$

In all cases therefore in which the Fourier series of

$$g(u) = \tfrac{1}{4}[f(x+2u) - f(x-2u)] \operatorname{cosec} u$$

converges for $u = 0$, the derived series of the Fourier series of $f(x)$ converges in the Cesàro manner, index unity, to the same value as the former series. And when the former series converges $(C, p-1)$, the latter series converges (Cp).

22. There are accordingly four cases to be specially mentioned in which the derived series of the Fourier series of $f(x)$ converges $(C1)$ at the point x.

(A) *If $[f(x+t)-f(x-t)]/t$ is a function of t which has bounded variation in an interval containing the origin $t = 0$.*

In this case

$$g(u) - [f(x+2u)-f(x-2u)]/4u = \tfrac{1}{2}\left(\frac{u-\sin u}{\sin u}\right)[f(x+2u)-f(x-2u)]/2u,$$

and, being the product of a function of u which is an integral by a function of bounded variation, is a function of bounded variation in the interval mentioned, and has the unique limit zero at $u = 0$. Therefore the left-hand side is a function of u whose Fourier series converges at $u = 0$ to zero, so that the Fourier series of $g(u)$ converges to the same value as that of $[f(x+2u)-f(x-2u)]/4u$, the convergence of the latter series being ensured by the hypothesis made, and its sum being

$$\operatorname*{Lt}_{u\to 0}\,[f(x+2u)-f(x-2u)]/4u,$$

which is the generalised differential coefficient of $f(x)$ at the point x.

Thus, if $[f(x+t)-f(x-t)]/t$ has bounded variation in an interval containing the origin $t = 0$, the derived series of the Fourier series of $f(x)$ converges to the generalised differential coefficient of $f(x)$.

(B) *If for some value of the constant K the function (of u)*

$$[f(x+u)-f(x-u)-2Ku]/u^2$$

is absolutely integrable (summable) in a certain interval containing $u = 0$.

In this case

$$\frac{g(u)-K}{u} = \frac{f(x+2u)-f(x-2u)-4K\sin u}{4u\sin u}$$

$$= \frac{f(x+2u)-f(x-2u)-4Ku}{4u^2}\,\frac{u}{\sin u} + K\,\frac{u-\sin u}{u\sin u},$$

and is absolutely integrable in the interval mentioned. Therefore, by the condition of § 9, the Fourier series of $g(u)$ converges at $u = 0$ to the value K.

Thus if, for some value of the constant K,

$$[f(x+u)-f(x-u)-2Ku]/u^2$$

is absolutely integrable in a certain interval containing $u = 0$, the derived series of the Fourier series of $f(x)$ converges at the point x to the value K.

(C) *If* $f(x)$ *has a generalised differential coefficient at the point* x, *and*

$$\frac{1}{h}\int_0^h |\,d[f(x+u)-f(x-u)]\,|$$

is a bounded function of h *in some interval containing* $h = 0$.

Putting $\qquad \phi(u) = [f(x+2u)-f(x-2u)]/4u,$

$\phi(u)$ has a unique limit when $u \to 0$, namely, the generalised differential coefficient of $f(x)$. Also

$$\frac{1}{h}\int_0^h |\,d[u\phi(u)]| = \frac{1}{4h}\int_0^h |\,d[f(x+2u)-f(x-2u)]\,|$$

$$= \frac{1}{4h}\int_0^{2h} |\,d[f(x+u)-f(x-u)]\,|,$$

and is therefore a bounded function of h in some interval containing $h = 0$.

Thus both the conditions involved in my test (§ 5) for the convergence of the Fourier series of $\phi(u)$ at $u = 0$ are satisfied. It follows that this series converges at $u = 0$ to the generalised differential coefficient of $f(x)$ at the origin.

But $$g(u)-\phi(u) = \tfrac{1}{4}\left(\frac{u-\sin u}{u \sin u}\right)[f(x+2u)-f(x-2u)],$$

which is a function of u having the unique limit zero at the origin, and is also a function of bounded variation in an interval containing the origin, since $[f(x+2u)-f(x-2u)]$ is a function of bounded variation in virtue of the second of the given conditions, and $\dfrac{u-\sin u}{u \sin u}$ is an integral. Thus the Fourier series of $g(u)-\phi(u)$ converges to zero at the origin. It follows that the Fourier series of $g(u)$ converges at the origin to the generalised differential coefficient of $f(x)$. *Thus in this case the derived series of the Fourier series of $f(x)$ converges at the point x to the generalised differential coefficient, that is to*

$$\operatorname*{Lt}_{h\to 0}\ [f(x+h)-f(x-h)]/2h.$$

As an immediate corollary from the result (C), we have, in particular, the two following results, which bear the same relation to (C) that Cor. 2 and Cor. 3 of § 5 do to the theorem of that article:—

If $f(x)$ has a generalised differential coefficient at the origin, and one of the derivates of $f(x)$ [or more generally those of $\{f(x+u)-f(x-u)\}$ with respect to u], is known to be, in a certain neighbourhood of the point,

(*a*) *bounded,*

or (*b*) *summable, and, except possibly at a countable set of points, finite,*

then the derived series of the Fourier series of $f(x)$ converges at the point x to the generalised differential coefficient of $f(x)$.

As a particular case we have the theorem of § 13.

(D) *If* $\dfrac{1}{u}\displaystyle\int_0^u \frac{f(x+u)-f(x-u)}{u}\,du$ *is a function of u of bounded variation in an interval containing the origin.*

In this case, writing

$$\psi(u) = [f(x+2u)-f(x-2u)]/4u,$$

we have $$\frac{1}{u}\int_0^u \psi(u)\,du = \frac{1}{2u}\int_0^{2u} \frac{f(x+u)-f(x-u)}{u}\,du,$$

a function of u of bounded variation. Thus, by de la Vallée Poussin's test, the Fourier series of $\phi(u)$ converges at $u=0$ to $\underset{u\to 0}{\text{Lt}} \frac{1}{u}\int_0^u \phi(u)du$.

In order therefore to prove that the Fourier series of $g(u)$ converges to the same value, it is only necessary to prove that

$$\frac{1}{u}\int_0^u [g(u)-\phi(u)]\,du$$

is a function of bounded variation whose limit when $u=0$ is zero.

Now

$$\frac{1}{u}\int_0^u [g(u)-\psi(u)]\,du = \frac{1}{4u}\int_0^u [f(x+2u)-f(x-2u)]\left(\frac{u-\sin u}{u\sin u}\right)du$$

$$= \frac{1}{2u}\int_0^u q(u)\,d[F(x+2u)+F(x-2u)],$$

where $F(x)$ is the integral of $f(x)$, and $q(u)=\dfrac{u-\sin u}{u\sin u}$ is a function such that $q(u)/u$, as well as the differential coefficient $q'(u)$, is an integral.

Integrating by parts, we get now

$$\frac{1}{2u}q(u)\,[F(x+2u)+F(x-2u)]-\frac{1}{2u}\int_0^u q'(u)\,[F(x+2u)+F(x-2u)]\,du,$$

an expression of which the first term is the product of two integrals, and is therefore a function of bounded variation, and the second term is also a function of bounded variation, since the integrand is a function of bounded variation, so that the integral divided by u is also a function of bounded variation.

Also, when $u\to 0$, the first term of the expression vanishes, since $q(u)$ has zero as limit, and the second term approaches half the value at $u=0$ of the integrand, since the integrand is continuous; but $q'(0)=0$, so that this second term also approaches zero. Thus we have proved that $\frac{1}{u}\int_0^u [g(u)-\phi(u)]\,du$ is a function of bounded variation whose limit when $u=0$ is zero. This proves that *in this case the derived series of the Fourier series of $f(x)$ converges to*

$$\underset{u\to 0}{\text{Lt}}\ \frac{1}{u}\int_0^u \frac{f(x+u)-f(x-u)}{u}\,du.$$

COR.—*If $F(x)$ be the integral of $f(x)$, it is sufficient for the conver-*

gence (*C*1) *of the first derived series of the Fourier series of* $f(x)$ *that* $[F(x+u)+F(x-u)-2F(x)]/u^2$ *should be a function of bounded variation.*

23. The formula (III) gives us corresponding conditions for the convergence (*Cp*) of the derived series of the Fourier series of $f(x)$. We merely have to write down conditions that the Fourier series of

$$g(u) = \tfrac{1}{4}[f(x+2u)-f(x-2u)]\operatorname{cosec} u,$$

should converge $(C, p-1)$ at $u = 0$.

Thus, for instance, using the conditions for the Cesàro convergence of a Fourier series contained in § 6 from my new condition for the convergence of a Fourier series, we have the following conditions :—

The first derived series of the Fourier series of $f(x)$ *converges* (*C*2) *at the point* x, *if* $\phi_2(+0)$ *exists, where*

$$\phi_2(u) = \frac{1}{u}\int_0^u \phi_1(t)\,dt = \frac{1}{u}\int_0^u \frac{1}{2t}[f(x+t)-f(x-t)]\,dt$$

$$= \frac{1}{u}\int_0^1 \tfrac{1}{2}[f(x+ut)-f(x-ut)]\,dt,$$

and if

$$\frac{1}{u}\int_0^u |\,d[t\phi_2(t)]| = \frac{1}{u}\int_0^u |\,\phi_1(t)\,dt\,| = \frac{1}{u}\int_0^u \left|\,\frac{1}{2t}[f(x+t)-f(x-t)]\,dt\,\right|$$

is a bounded function of u *in a certain interval containing* $u = 0$.

Indeed if these conditions are satisfied, they are also satisfied when we change u into $2u$, so that the Fourier series of $\psi_1(2u)$ converges (*C*1).

But, as we saw in § 22, $g(u)-\psi_1(2u)$ is a function of bounded variation of u which is continuous at the origin and has there the value zero, so that its Fourier series converges at $u = 0$ to zero. Hence the Fourier series of $g(u)$ converges (*C*1) at $u = 0$, whence the required result is also true.

Similarly, we have the more general condition, as follows :—

The first derived series of the Fourier series of $f(x)$ *converges* (*Cp*) *at the point* x, *if* $\psi_p(+0)$ *exists, where*

$$\psi_p(u) = \frac{1}{u}\int_0^1\int_0^1 \cdots \tfrac{1}{2}[f(x+ut_1t_2\ldots t_{p-1})-f(x-ut_1t_2\ldots t_{p-1})]\,dt_1\,dt_2\ldots dt_{p-1},$$

and if $\frac{1}{u}\int_0^u |\psi_{p-1}(t)dt|$ *is a bounded function of* u *in a certain interval containing* $u=0$.

24. The discussion of the convergence $(C2)$ of the second derived series, and the convergence (Cp) of the p-th derived series, may be undertaken on the same lines as have been followed in §§ 20–23 for the first derived series.

Let $f(x)$ be a periodic summable function of x, with period 2π, and let its Fourier series be written in the following form:—

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)/n^2 = \tfrac{1}{2}a_0 - \sum_{n=1} a_n/n^2.$$

Now let us write

$$g(u) = [f(x+2u)+f(x-2u)-2f(x)]/4\sin^2 u.$$

Then $g(u)$ is an even function of u, such that $g(\pi+u)=g(u)$, and therefore its Fourier series has no sine terms, and the cosines of the odd multiples of u are absent. We may therefore write

$$g(u) \sim \tfrac{1}{2}A_0 + \sum_{n=1} A_n \cos 2nu.$$

We then have, for $n \geqslant 0$,

$$\tfrac{1}{2}\pi(A_n - A_{n+1}) = \int_{-\pi}^{\pi} g(u)\,\tfrac{1}{2}[\cos 2nu - \cos 2(n+1)u]\,du$$

$$= \int_0^{\pi} 2g(u)\sin u \sin(2n+1)u\,du\,;$$

and therefore, for $n \geqslant 1$,

$$\tfrac{1}{2}\pi(A_{n-1} - 2A_n + A_{n+1})$$

$$= \int_0^{\pi} 2g(u)\sin u\,[\sin(2n-1)u - \sin(2n+1)u]\,du$$

$$= -\int_0^{\pi} [f(x+2u)+f(x-2u)-2f(x)]\cos 2nu\,du$$

$$= \sum_{m=1} \frac{1}{m^2}\int_0^{\pi} 2(a_m \cos mx + b_m \sin mx)(1-\cos 2mu)\cos nu\,du.$$

Hence $A_{n-1} - 2A_n + A_{n+1} = -2(a_n \cos nx + b_n \sin nx)/n^2 = 2a_n$.

We have then

$$\sum_{r=1}^{n} r^2 (A_{r-1} - 2A_r + A_{r+1}) = 2\sum_{r=1}^{n} a_r = 2s_n,$$

say. On the left the coefficient of A_0 is 1, of A_1 is 2, of A_n is

$$[(n-1)^2-n^2] = (n+1)^2-2,$$

and of A_{n+1} is n^2; and that of each of the remaining quantities A_r is

$$(r-1)^2-2r^2+(r+1)^2 = 2.$$

Thus our equation becomes, when we write

$$S_n = \tfrac{1}{2}A_0+A_1+\ldots+A_n,$$

$$2S_r-(r+1)^2 A_r+r^2 A_{r+1} = 2s_r. \qquad \text{(IV)}$$

Adding the equations (IV) for all values of r from 1 to n, we evidently get

$$2(S_1+S_2+\ldots+S_n)-4\sum_{r=1}^{n} rA_r+n^2A_{n+1} = 2(s_1+s_2+\ldots+s_n).$$

Since, however,

$$A_1+2A_2+\ldots+nA_n = (S_1-S_0)+2(S_2-S_1)+\ldots+n(S_n-S_{n-1})$$

$$= (n+1)S_n-\sum_{r=0}^{n} S_r,$$

we get, writing

$$C(S_n) = (S_1+S_2+\ldots+S_n)/n, \qquad C(s_n) = (s_1+s_2+\ldots+s_n)/n,$$

$$6C(S_n)+4S_0/n-4(n+1)S_n/n+nA_{n+1} = 2C(s_n). \qquad \text{(V)}$$

Proceeding to the limit, this becomes

$$\operatorname*{Lt}_{n\to\infty}[6C(S_n)-4S_n+nA_n] = 2\operatorname*{Lt}_{n\to\infty} C(s_n); \qquad \text{(V')}$$

and, operating on (V) in the Cesàro manner and proceeding to the limit, we get

$$\operatorname*{Lt}_{n\to\infty}[6(C2)(S_n)-5C(S_n)+S_n] = 2\operatorname*{Lt}_{n\to\infty}(C2)(s_n). \qquad \text{(VI)}$$

Generally*

$$\operatorname*{Lt}_{n\to\infty}[6(Cp)(S_n)-5(C,p-1)(S_n)+(C,p-2)(S_n)] = 2\operatorname*{Lt}_{n\to\infty}(Cp)(S_n); \qquad \text{(VII)}$$

that is, denoting the Cesàro summation, index p, of the second derived series of the Fourier series of $f(x)$ by t_p, and that of $g(u)$ for $u=0$ by T_p,

$$6T_p-5T_{p-1}+T_{p-2} = 2t_p. \qquad \text{(VII')}$$

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* As on p. 264, footnote, of "The Convergence of a Fourier Series and its Allied Series", this relation holds in the first instance for positive integral values of p, but is true when p is fractional also.

The corresponding form of (V'), is the following, in which the index zero denotes an ordinary summation :—

$$6T_1-4T_0+\operatorname*{Lt}_{n\to\infty} nA_n = t_0. \qquad (V'')$$

In all cases therefore in which the Fourier series of

$$g_2(u) = [f(x+2u)+f(x-2u)-2f(x)]/4\sin^2 u$$

converges for $u = 0$, *the second derived series of the Fourier series of* $f(x)$ *converges* $(C2)$ *to the same sum; also when the former series converges* $(C, p-2)$, *the latter series converges* (Cp).

Hence, as is evident, we have tests for the convergence $(C2)$ of the second derived series of the Fourier series of $f(x)$ at the point x exactly corresponding to those given in § 21 for the convergence $(C1)$ of the first derived series, and we have tests for the convergence $(C, p+2)$ of the second derived series, corresponding to the tests for the convergence $(C, p+1)$ of the first derived series.

25. The treatment of the third and higher derived series is precisely similar ; we only have to replace the function $g_2(u)$ used in the discussion of the second derived series by

$$g_3(u) = [f(x+2u)-f(x-2u)-\psi_1(x)\sin 2u]/\sin^3 u$$

in dealing with the third derived series, and by

$$g_4(u) = [f(x+2u)+f(x-2u)-2f(x)-\phi_1(x)\sin^2 u]/\sin^4 u$$

in dealing with the fourth derived series. Here ψ_1 and ϕ_1 are in the first instance such functions of x as render g_3 and g_4 summable functions of u. In consequence of the conditions introduced in the tests employed, these functions of x come to be—to a numerical factor *près*—the generalised first and second differential coefficient of $f(x)$, the generalised differential coefficients being accordingly required to exist.

More generally the auxiliary functions to be employed are the following :—

$$g_{2n+1}(u) = [f(x+2u)-f(x-2u)-\psi_1(x)\sin 2u-\psi_2(x)\sin^3 2u-\psi_3(x)\sin^5 2u \\ -\ldots-\psi_n(x)\sin^{2n-1} 2u]/\sin^{2n+1} u,$$

and

$$g_{2n}(u) = [f(x+2u)+f(x-2u)-2f(x)-\phi_1(x)\sin^2 u-\phi_2(x)\sin^4 u \\ -\ldots-\phi_{n-1}(x)\sin^{2n-2} u]/\sin^{2n} u.$$

26. Taking, for example, the case of the fourth derived series, we have, g_2 and g_4 having the meanings assigned to them in the preceding article,

$$g_4(u)\sin^2 u = g_2(u) - \tfrac{1}{4}\phi_1(x).$$

Hence, if $$g_{2r}(u) \sim \tfrac{1}{2}A_0^{2r} + \sum_{n=1}^{\infty} A_n^{2r}\cos 2nu,$$

we have
$$\begin{aligned} A_{n-1}^{(4)} - 2A_n^{(4)} + A_{n+1}^{(4)} &= -\frac{2}{\pi}\int_0^{\pi} 4g_4(u)\sin^2 u\cos 2nu\,du \\ &= -\frac{2}{\pi}\int_0^{\pi} 4\left[g_2(u) - \tfrac{1}{4}\phi_1(x)\right]\cos 2nu\,du \\ &= -\frac{2}{\pi}\int_0^{\pi} 4g_2(u)\cos 2nu\,du \quad (n \geqslant 1) \\ &= -4A_n^{(2)}. \end{aligned}$$

Since $$A_{n-1}^{(2)} - 2A_n^{(2)} + A_{n+1}^{(2)} = \frac{2}{n^4}(a_n\cos nx + b_n\sin nx) \quad (n \geqslant 1)$$

we get

$$A_{n-2}^{(4)} - 4A_{n-1}^{(4)} + 6A_n^{(4)} - 4A_{n+1}^{(4)} + A_{n+2}^{(4)} = -8(a_n\cos nx + b_n\sin nx)/n^4 \quad (n \geqslant 2).$$

In the general case in which $g_p(u)$ takes the place of $g_4(u)$, we have the corresponding formula

$$A_{n-p+1}^{(p)} - pA_{n-p+2}^{(p)} + \binom{p}{2}A_{n-p+3}^{(p)} - \ldots + (-)^r\binom{p}{r}A_{n-p+r+1}^{(p)} + \ldots$$
$$= M(a_n\cos nx + b_n\sin nx)/n^p,$$

the coefficients $\binom{p}{r}$ being the binomial coefficients, and M a numerical factor.

Multiplying up by n^p and making use of a process exactly analogous to that already used for $p = 1$ or $p = 2$, we find that the convergence (Cp) of the p-th derived series follows from the ordinary convergence of the Fourier series of $g_p(u)$ at $u = 0$, whenever this occurs.

A SYMMETRICAL CONDITION FOR CO-APOLAR TRIADS ON A CUBIC CURVE

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1. *Introduction.*

Two triads of points ABC, DEF situated on a cubic curve are said to be "co-apolar" if any point P on the curve possesses the property that the pencils $P[ABC]$ and $P[DEF]$ are apolar. I have shewn in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243, that the triads ABC, DEF cannot be arbitrarily chosen on the cubic, but must each be of a special type to which the name "apolar triad" was given. An apolar triad is such that the bi-polar line of any two of its constituent points passes through its third point. In the foregoing paper, I found the necessary and sufficient condition that two apolar triads should be co-apolar, but the form of the condition was unsymmetrical with respect to the two given triads. Inasmuch as the problem under investigation is essentially one of symmetry, it is desirable to obtain a solution in which the given triads are similarly involved. Such is the purpose of the present communication.

The essence of the method lies in projecting the two given triads through a suitably chosen point of the curve on to the curve again so as to obtain two triads lying on the same conic, and thereafter to express the required condition in terms of the covariant geometry of two triads of points on a conic.

2. *A Pencil of Cubic Curves.*

Let ABC and DEF be two triads on a cubic curve, and let them be projected through a point T of the curve on to the curve again, thus giving the triads $A'B'C'$ and $D'E'F'$. It was proved in the above-mentioned paper that, if ABC and DEF be two apolar triads which stand in the co-apolar relation, then $A'B'C'$ and $D'E'F'$ will also be two apolar

triads co-apolarly related. It will be convenient if we choose T so that $A'B'C'$, $D'E'F'$ lie on the same conic Γ. The positions of T are easily found in terms of elliptic parameters: for, if A, B, &c., represent the elliptic parameters of the points A, B, &c., we have $A+T+A' \equiv 0$, &c., and $\Sigma A' \equiv 0$, whence $\Sigma A+6T = 0$, giving thirty-six positions of T. Let us consider next the pencil Ψ of cubics circumscribing the triangles $A'B'C'$, $D'E'F'$ and apolar to them. One cubic of the system is the degenerate cubic consisting of Γ and L, where L is the line joining the poles of Γ with respect to the triangles $A'B'C'$ and $D'E'F'$. It is therefore evident that the nine "associated" points, common to all the members of the pencil Ψ, consist of the triads $A'B'C'$, $D'E'F'$ and three other points $U'V'W'$, which lie on L, since $A'B'C'$, $D'E'F'$ lie on the conic Γ. Furthermore, I have proved in the *Proceedings of the Edinburgh Mathematical Society*, Vol. 30, Session 1911–12, that if a pencil of cubics circumscribe two triangles and be apolar to them, the remaining three points common to all the curves of the system form also a triangle apolar to every member of the system. In the above case, therefore, since U', V', W' form a linear triad apolar to every member of the Ψ-pencil, we find that the line L joining U', V', W' must touch the Cayleyan of every member of the Ψ-pencil, as was proved in the aforementioned communication to the London Mathematical Society, for the case of linear apolar triads. The tangents at U', V', W' to any member of the Ψ-pencil are therefore concurrent; and the locus of the point of concurrency is a straight line M, since a (1, 1) correspondence exists between the tangents at U', V' (or W'), and $U'V'$ (regarded as a ray from U') corresponds to $V'U'$ (regarded as a ray from V') for the case of the cubic ΓL. It is easy to identify the line M by considering two particular members of the Ψ-system, viz. the apolar loci of $[A'B'C', U'V'W']$ and $[D'E'F', U'V'W']$. I proved (*Proc. London Math. Soc., loc. cit.*) that the three tangents at U', V', W' to the above apolar loci meet at the poles of the line L with respect to the triangles $A'B'C'$, $D'E'F'$ respectively. Hence *M is the line joining the poles of L with respect to the triangles $A'B'C'$, $D'E'F'$.* It is noteworthy that the poles of L with respect to these two triangles lie on the conic Γ, since the poles of Γ with respect to the triangles lie on L.

3. *The Required Condition.*

We thus see that, in virtue of the fact that the tangents at $U'V'W'$ are concurrent on M for every member of the Ψ-pencil of cubics, we have established a (1, 1) correspondence between the members of the Ψ-system

and the points on the line M. We wish to obtain now the point Q on M that corresponds to the apolar locus of $[A'B'C', D'E'F']$, *i.e.* the point on the line M at which meet the tangents at U', V', W' to the above apolar locus.

Take the conic Γ in the form $x = t^2$, $y = 1$, $z = 2t$, where the tangents at the Hessian points of $A'B'C'$ and the line joining these points form the triangle of reference. Also let us take $D'E'F'$ in the form

$$a_0 t^3 + 3a_1 t^2 + 3a_2 t + a_3 = 0. \tag{1}$$

Let H and K be the poles of the conic Γ with respect to the triangles $A'B'C'$ and $D'E'F'$ respectively.

A cubic of the Ψ-pencil other than ΓL is

$$\Delta \equiv 8a_0x^3 + 8a_3y^3 + (a_0+a_3)z^3 + 12a_1x^2z + 12a_2y^2z + 6a_2z^2x + 6a_1z^2y = 0. \tag{2}$$

The apolar locus of $A'B'C'$ and $D'E'F'$ is easily found to be

$$(a_0 - a_3)\Delta = 18\Gamma L, \tag{3}$$

where $$\Gamma \equiv z^2 - 4xy$$

and $$L \equiv (a_0a_2 - a_1^2)x - (a_1a_3 - a_2^2)y.$$

Since the tangents at U', V', W' meet at Q, we see that L is part of the polar conic of Q with respect to the above apolar locus. Hence the polar conic of Q passes through H, shewing also that the polar line of H passes through Q. Q is therefore defined as the point of intersection of the polar line of H with M. Now the polar line of H with respect to (3) is

$$(a_0 - a_3)\{2a_2x + 2a_1y + (a_0+a_3)z\} = 6\{(a_0a_2 - a_1^2)x - (a_1a_3 - a_2^2)y\}. \tag{4}$$

The symmetry of this result (4) with regard to the invariants and covariants involved shews that it is also the polar line of K.

It is possible to identify geometrically the line (4) in the following way. There are four points on Γ (viz., the Jacobian-tetrad) whose polar conics with respect to the triangles $A'B'C'$ and $D'E'F'$ intersect on Γ itself. The conic locus J passing through these four points and apolar to the conic envelope Γ is

$$J \equiv 6a_1x^2 - 6a_2y^2 + (a_3 - a_0)z^2 - 6a_1yz + 6a_2zx + 2(a_3 - a_0)xy = 0. \tag{5}$$

Also there are two points on Γ whose polar lines with respect to the triangles $A'B'C'$, $D'E'F'$ are conjugate with respect to Γ. If the tangents

at these two points meet at O, we have

$$O \equiv \{a_1, a_2, -(a_0+a_3)\}. \qquad (6)$$

By means of (5) and (6), it is easy to deduce that *the polar line of H (or K) with respect to the apolar locus of $(A'B'C', D'E'F')$ is the polar of the point O with respect to the conic J.*

The point on the line M which corresponds to the apolar locus of $[A'B'C', D'E'F']$ according to the above method has therefore been identified geometrically.

We have therefore this final result :—

Let two apolar triads ABC, DEF be projected through a suitable point T into the apolar triads $A'B'C'$, $D'E'F'$, so that the six points thus obtained lie on the same conic Γ. Let L be the line joining the poles H, K of Γ with respect to the triangles $A'B'C'$, $D'E'F'$, and let the cubic intersect L in U', V', W'. Let M be the line joining the poles of L with respect to the triangles $A'B'C'$, $D'E'F'$. Let the J-conic and the O-point be defined as in (5) *and* (6) *above with respect to the conic Γ and the triads $A'B'C'$, $D'E'F'$ lying on Γ. Then the necessary and sufficient condition that the apolar triads ABC, DEF be co-apolar is that the point of concurrency of the tangents to the cubic at U', V', W' be at the point of intersection of the polar of O with respect to the conic J, and M the line joining the poles of L with respect to the triangles $A'B'C'$ and $D'E'F'$.*

THE INTEGRAL FORMULA FOR GENERALISED LEGENDRE FUNCTIONS

By G. N. Watson.

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1. The formulæ

$$\int_{-1}^{1} \{P_n(\mu)\}^2 d\mu = \frac{2}{2n+1}, \quad \int_{-1}^{1} \{P_n^m(\mu)\}^2 d\mu = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!},$$

valid when n and m are positive integers, are fundamental in various branches of the theory of Legendre functions. It is natural to enquire whether these formulæ can be generalised, so as to yield a result which shall be valid when n and m are no longer restricted to be positive integers, but may be any numbers, real or complex.

The generalisation has been given by Barnes* in the two forms:

$$\int_{(0)}^{(1+)} \{P_n^m(z)\}^2 dz$$

$$= \frac{2i\sin m\pi}{2n+1}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}$$

$$-\frac{i\sin m\pi}{\Gamma(-n-m)\,\Gamma(n-m+1)}\sum_{t=0}^{\infty}\frac{\Gamma(n-m+1+t)\,\Gamma(n+m+1+t)}{t!\,\Gamma(2n+2+t)(t+n+\frac{1}{2})}$$

$$= \frac{2i\sin m\pi}{\Gamma(2-2m)}\,{}_3F_2(n-m+1,\ -n-m,\ \tfrac{1}{2}-m;\ 1-2m,\ \tfrac{3}{2}-m;\ 1),$$

the contour starting from the origin and returning to it after a positive circuit round the point 1. Incidentally, Barnes obtained the relation indicated between the generalised hypergeometric functions involved in these results.

It does not seem to have been noticed that the contour integral can be evaluated in terms of what are usually regarded as known functions of

* *Quarterly Journal*, Vol. 39 (1907), pp. 186–188.

analysis, namely Gamma functions and their logarithmic derivates. In view of the importance of the definite integrals in physical problems, as well as from the purely mathematical aspect, it seems desirable to place on record the generalised formula

$$\int_{(0)}^{(1+)} \{P_n^m(z)\}^2 dz = \frac{i \sin m\pi \sin (n+m)\pi}{(2n+1)\pi} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}$$

$$\times \{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(-\tfrac{1}{2}n-\tfrac{1}{2}m)-\psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)\}.$$

Needless to say, this can be reduced to the definite integral formula when n and m are made integers. It should be remarked that the formula is *not* a case of Appell's formula* for

$$\int_0^1 x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma} \{F(\alpha, \beta; \gamma; x)\}^2 dx,$$

since 0 and 1 are singular points of this hypergeometric function, while the origin is not a singular point of the hypergeometric function of $\frac{1}{2}(1-z)$ in terms of which $P_n^m(z)$ is defined. The method which I employ is rather different from that employed by Appell, as it makes use of the derivates of the Legendre functions with regard to their degree n. These derivates seem to have made their first appearance in two papers† communicated to the Society some years ago.

2. If we differentiate the equation

$$(1-z^2)\frac{d^2P_n^m(z)}{dz^2} - 2z\frac{dP_n^m(z)}{dz} + \left\{n(n+1) - \frac{m^2}{1-z^2}\right\} P_n^m(z) = 0,$$

with regard to n, we get

$$(1-z^2)\frac{d^2}{dz^2}\frac{\partial P_n^m(z)}{\partial n} - 2z\frac{d}{dz}\frac{\partial P_n^m(z)}{\partial n} + \left\{n(n+1) - \frac{m^2}{1-z^2}\right\}\frac{\partial P_n^m(z)}{\partial n}$$

$$+(2n+1)\, P_n^m(z) = 0.$$

Multiply these equations by $\partial P_n^m(z)/\partial n$ and $P_n^m(z)$ respectively, subtract

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* *Comptes Rendus*, Vol. 87 (1878), p. 874; Vol. 89 (1879), pp. 31–33. A paper by Callandreau, *Comptes Rendus*, Vol. 89 (1879), p. 90, gives an integral involving the product $F(\alpha, \beta; \gamma; x)\, F(\alpha', \beta'; \gamma'; z)$.

† Bromwich, *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1912), pp. 100–125. Watson, *ibid.*, *Abstracts*, pp. viii–xii.

and then integrate, and we find that

$$(2n+1)\int \{P_n^m(z)\}^2\,dz = \left[(1-z^2)\left\{\frac{\partial P_n^m(z)}{\partial n}\,\frac{dP_n^m(z)}{dz} - P_n^m(z)\,\frac{\partial}{\partial n}\,\frac{dP_n^m(z)}{dz}\right\}\right];$$

it should be noticed in passing that, for the values of z under consideration, it is permissible to interchange the order of differentiations with regard to n and z.

Now, near $z = 0$, we have*

$$P_n^m(z) = 2^m(z^2-1)^{-\frac{1}{2}m}\left\{\frac{\Gamma(\frac{1}{2})\,F(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2},\ -\frac{1}{2}n-\frac{1}{2}m;\ \frac{1}{2};\ z^2)}{\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)\,\Gamma(\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m)}\right.$$

$$\left. + \frac{z\Gamma(-\frac{1}{2})\,F(\frac{1}{2}n-\frac{1}{2}m+1,\ \frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m;\ \frac{3}{2};\ z^2)}{\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})\,\Gamma(-\frac{1}{2}n-\frac{1}{2}m)}\right\};$$

and, if we make the usual cut from $+1$ to $-\infty$, $\arg(z^2-1)$ is $-\pi$ at the beginning of the contour and $+\pi$ at the end of the contour.

Hence

$$P_n^m(0) = 2^m e^{\pm\frac{1}{2}m\pi i}\,\Gamma(\tfrac{1}{2})/\{\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+1)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)\},$$

$$\left\{\frac{dP_n^m(z)}{dz}\right\}_{z=0} = 2^m e^{\pm\frac{1}{2}m\pi i}\Gamma(-\tfrac{1}{2})/\{\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}n-\tfrac{1}{2}m)\}.$$

Differentiating with regard to n when necessary, and substituting, we at once get

$$(2n+1)\int_{(0)}^{(1+)} \{P_n^m(z)\}^2\,dz$$

$$= \frac{2^{2m}(e^{-m\pi i}-e^{m\pi i})\,\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)\,\Gamma(\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m)\,\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})\,\Gamma(-\frac{1}{2}n-\frac{1}{2}m)}$$

$$\times\{-\tfrac{1}{2}\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)+\tfrac{1}{2}\psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)+\tfrac{1}{2}\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})$$

$$-\tfrac{1}{2}\psi(-\tfrac{1}{2}n-\tfrac{1}{2}m)\}.$$

Now, by the duplication formula for the Gamma function,

$$\frac{2^{2m}(e^{-m\pi i}-e^{m\pi i})\,\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)\,\Gamma(-\frac{1}{2}n-\frac{1}{2}m)\,\Gamma(\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m)}$$

$$= -2\pi^{-1}\,i\,\sin m\pi\,\sin(n+m)\,\pi\,\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)},$$

* Hobson, *Phil. Trans.*, 187 (A) (1896), p. 469; Barnes, *loc. cit.*, p. 121.

and so we have the result

$$\int_{(0)}^{(1+)} \{P_n^m(z)\}^2 dz = \frac{i \sin m\pi \sin (n+m)\pi}{(2n+1)\pi} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}$$

$$\times \{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(-\tfrac{1}{2}n-\tfrac{1}{2}m)-\psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)\}.$$

This is in agreement with a result given by Barnes (*loc. cit.*, p. 189) in the limiting case when $n+m$ is made a positive integer.

3. If the sign of m be changed in the result of § 2, we get the corresponding integral involving $P_n^{-m}(z)$; also, since the product $P_n^m(z)\,P_n^{-m}(z)$ is analytic at $z=1$, we have

$$\int_{(0)}^{(1+)} P_n^m(z)\,P_n^{-m}(z)\,dz = 0.$$

Further, since*

$$2Q_n^m(z)\,\Gamma(-m-n)\,\pi^{-2} \sin m\pi \sin n\pi = \frac{P_n^{-m}(z)}{\Gamma(n-m+1)} - \frac{P_n^m(z)}{\Gamma(n+m+1)},$$

we have

$$\{2\Gamma(-m-n)\,\pi^{-2} \sin m\pi \sin n\pi\}^2 \int_{(0)}^{(1+)} \{Q_n^m(z)\}^2 dz$$

$$= \frac{\{(2n+1)\pi\}^{-1}\, i \sin m\pi}{\Gamma(n-m+1)\,\Gamma(n+m+1)}$$

$$\times [\sin(n+m)\pi \{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(-\tfrac{1}{2}n-\tfrac{1}{2}m)$$

$$-\psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)\}$$

$$-\sin(n-m)\pi \{\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2})+\psi(-\tfrac{1}{2}n+\tfrac{1}{2}m)$$

$$-\psi(\tfrac{1}{2}-\tfrac{1}{2}n+\tfrac{1}{2}m)\}].$$

Using the relation $\psi(1-x)-\psi(x) = \pi \cot \pi x$,

* This is Barnes' definition of $Q_n^m(z)$, (*loc. cit.*, p. 100), not Hobson's (*loc. cit.*, p. 456).

we may write the expression in [ ] in the form

$$\sin(n+m)\pi\{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(1+\tfrac{1}{2}n+\tfrac{1}{2}m)$$

$$-\psi(\tfrac{1}{2}+\tfrac{1}{2}n+\tfrac{1}{2}m)\}$$

$$-\sin(n-m)\pi\{\psi(1+\tfrac{1}{2}n+\tfrac{1}{2}m)-\psi(\tfrac{1}{2}+\tfrac{1}{2}n+\tfrac{1}{2}m)+\psi(1+\tfrac{1}{2}n-\tfrac{1}{2}m)$$

$$-\psi(\tfrac{1}{2}+\tfrac{1}{2}n-\tfrac{1}{2}m)\}$$

$$+\sin(n+m)\pi\{\pi\cot\tfrac{1}{2}(n+m)\pi-\pi\cot\tfrac{1}{2}(n+m-1)\pi\}$$

$$-\sin(n-m)\pi\{\pi\cot\tfrac{1}{2}(n-m)\pi-\pi\cot\tfrac{1}{2}(n-m-1)\pi\},$$

and those terms which involve trigonometrical functions only happen to cancel.

We thus obtain the formula

$$\int_{(0)}^{(1+)}\{Q_n^m(z)\}^2dz$$

$$=\tfrac{1}{2}\pi i\,\frac{\sin^2(n+m)\pi\cos n\pi}{(2n+1)\sin^2 n\pi}\,\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}$$

$$\times\{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2})\}.$$

Comparing this with Barnes' formula (*loc. cit.*, p. 194), we see that

$$(4n+2)\sum_{t=0}^{\infty}\frac{\Gamma(n+m+1+t)\,\Gamma(n-m+1+t)}{t!\,\Gamma(2n+2+t)(2t+2n+1)}$$

$$=\{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})\}.$$

This result affords an interesting special case in which the sum of a series of the type ${}_3F_2(\alpha, \beta, \gamma;\ \epsilon, \theta;\ 1)$ can be obtained in finite terms.

4. If we differentiate the generalised equation of Legendre's type with regard to m, we obtain the result

$$2m\int\frac{\{P_n^m(z)\}^2}{z^2-1}dz=\left[(1-z^2)\left\{\frac{\partial P_n^m(z)}{\partial m}\,\frac{dP_n^m(z)}{dz}-P_n^m(z)\,\frac{\partial}{\partial m}\,\frac{P_n(z)}{dz}\right\}\right]$$

in place of the formula given in § 2.

At $z = 0$, the integrated part is equal to

$$\frac{2^{2m} e^{\pm m\pi i} \Gamma(\frac{1}{2}) \Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}n-\frac{1}{2}m+1) \Gamma(\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m) \Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2}) \Gamma(-\frac{1}{2}n-\frac{1}{2}m)}$$

$$\times \{\log 2 \pm \tfrac{1}{2}\pi i + \tfrac{1}{2}\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1) + \psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)$$

$$-\log 2 \mp \tfrac{1}{2}\pi i - \psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2}) - \tfrac{1}{2}\psi(-\tfrac{1}{2}n-\tfrac{1}{2}m)\},$$

and so we have

$$\int_{(0)}^{(1+)} \frac{\{P_n^m(z)\}^2}{z^2-1}\, dz$$

$$= \frac{i \sin m\pi \sin(n+m)\pi}{2m\pi} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}$$

$$\times \{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2}) - \psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1) + \psi(-\tfrac{1}{2}n-\tfrac{1}{2}m) - \psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)\}.$$

By the methods of § 3, we can obtain the corresponding integral involving $\{Q_n^m(z)\}^2$.

Comparing the results of § 3 and § 4, we obtain the peculiar formula

$$\int_{(0}^{(1+)} \{Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(z)\}^2\, dz = \frac{\pi^3}{2\{\Gamma(m+n+1)\cos m\pi\}^2} \int_{(0)}^{(1+)} \frac{\{P_n^m(z)\}^2}{z^2-1}\, dz.$$

THE CLASSIFICATION OF RATIONAL APPROXIMATIONS

By J. H. GRACE.

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Introduction.

It was proved by Hermite* that, if θ is any incommensurable number, there exists an infinite number of rational approximations x/y such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{y^2\sqrt{3}},$$

and that for two consecutive approximations x/y, x'/y' of this class

$$xy' - x'y = \pm 1.$$

The same theorem has been proved by Minkowski† when $\sqrt{3}$ is replaced by 2.

The main object of this paper is to prove that, *if*

$$1\tfrac{1}{2} \leqslant k \leqslant 2,$$

then for two consecutive approximations that satisfy the condition

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2},$$

we have

$$xy' - x'y = \pm 1. \qquad \text{(A)}$$

If k is outside the limits stated, this last relation need not be satisfied by two consecutive approximations (§ 7).

What is alluded to as classification means that, when θ is given, a class of approximations consists of those satisfying

$$| x/y - \theta | < 1/ky^2,$$

* *Œuvres*, Vol. 1, p. 168. Cf. also the letters to Jacobi.

† *Ges. Abhandlungen*, Vol. 1, p. 320; *Math. Ann.*, Vol. 54, p. 91.

where k is fixed. The class is infinite for all θ's when $k \leqslant \sqrt{5}$.* When approximations are said to be arranged in order, it is order of increasing complexity, *i.e.* increasing values of y, that is meant.

1. We shall first prove that, if

$$|x/y - \theta| < 1/y^2,$$

and the fraction x/y is in its lowest terms, then it is a convergent to the continued fraction for θ.

In fact, express x/y as a continued fraction, and arrange so that the penultimate convergent x'/y' errs from x/y in the same direction that θ differs from x/y.

Since $$|x/y - x'/y'| = 1/yy',$$

it is clear that θ lies between the two fractions; we can therefore write

$$\theta = \frac{px+x'}{py+y'} \quad (p > 0);$$

and hence, if $$\frac{x}{y} = c_0 + \frac{1}{c_1} + \cdots \frac{1}{c_n},$$

or $$(c_0;\ c_1, c_2, \ldots, c_n),$$

we have $$\theta = (c_0;\ c_1, c_2, \ldots, c_n, p).\dagger$$

If $p > 1$, the C.F. for θ agrees with that for x/y as far as c_n; if $p < 1$, the C.F. for θ only differs in that c_n is to be increased by the integral part of $1/p$. In the first case, x/y is a principal convergent to the C.F. for θ, and in the second it is an auxiliary convergent.

2. Let the C.F. for θ be

$$(a_0;\ a_1, a_2, \ldots, a_n, \ldots),$$

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* Hurwitz, *Math. Ann.*, Vol. 39, p. 279, and Markoff, *ibid.*, Vol. 15, p. 384. The approximations of Hermite do not include all that satisfy the condition indicated. What they are and what they are not has been elegantly determined by Humbert, *Comptes Rendus*, Dec. 1915. Cf. Minkowski, *Ges. Abhandlungen*, Vol. 1, p. 279. The case $\Omega = 1$ is here referred to.

† We also write

$$(c_0;\ c_1, c_2, c_3, \ldots) \quad \text{for} \quad c_0 + \frac{1}{c_1} + \frac{1}{c_2} + \cdots$$

and $$(c_1, c_2, c_3, \ldots) \quad \text{for} \quad \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots.$$

and denote the convergent corresponding to a_n by p_n/q_n; and write

$$\omega_n = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots).$$

Then
$$\theta = \frac{\omega_n p_n + p_{n-1}}{\omega_n q_n + q_{n-1}},$$

and
$$\left| \frac{p_n}{q_n} - \theta \right| = \frac{1}{\lambda_n q_n^2},$$

where $\lambda_n = \omega_n + \dfrac{q_{n-1}}{q_n} = (a_{n+1};\ a_{n+2},\ a_{n+3},\ \ldots) + (a_n,\ a_{n-1},\ a_{n-2},\ \ldots).$ (2 . 1)

Next consider an auxiliary (or intermediate) convergent p/q, where

$$p = \alpha p_{n-1} + p_{n-2}, \qquad q = \alpha q_{n-1} + q_{n-2},$$

α being an integer less than a_n. It is easy to see that

$$\left| \frac{p}{q} - \theta \right| = \frac{1}{\lambda q^2},$$

where
$$\lambda = (\beta,\ a_{n+1},\ a_{n+2},\ \ldots) + (\alpha,\ a_{n-1},\ a_{n-2}), \qquad (2 . 2)$$

and
$$\alpha + \beta = a_n.$$

In fact, we have
$$\theta = \frac{\eta p + p_{n-1}}{\eta q + q_{n-1}},$$

where
$$\eta = (\beta,\ a_{n+1},\ a_{n+2},\ \ldots),$$

and so
$$\frac{p}{q} - \theta = \frac{1}{q(\eta q + q_{n-1})} = \frac{1}{q^2(\eta + q_{n-1}/q)};$$

and since
$$\frac{q_{n-1}}{q} = (\alpha,\ a_{n-1},\ a_{n-2},\ \ldots),$$

the result follows.

If p/q is really an auxiliary (and not a principal) convergent, α, β are both positive. Hence λ is in all cases less than 2, and less than 1 unless either α or β is unity. Thus :—

The approximations to θ which belong to the class defined by

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{y^2},$$

are all included among the principal convergents for θ and the auxiliary

convergents adjacent to them. The former must satisfy the condition and the latter may do.

The formulæ (2 . 1) and (2 . 2) are fundamental.

3. If a convergent intermediate between p_{n-1}/q_{n-1} and p_n/q_n satisfies the condition

$$\left|\frac{p}{q}-\theta\right|<\frac{1}{kq^2},$$

then p_n/q_n also satisfies the condition. For

$$\left|\frac{p}{q}-\theta\right|=\frac{1}{\lambda q^2},$$

$$\left|\frac{p_n}{q_n}-\theta\right|=\frac{1}{\lambda_n q_n^2},$$

and we have only to see that $\lambda_n>\lambda$.

We have $$\lambda=\frac{1}{\beta+x}+\frac{1}{\alpha+y} \quad (x,\ y<1),$$

$$\lambda_n=\frac{1}{x}+\frac{1}{a_n+y},$$

$$\lambda_n-\lambda=\frac{\beta}{x(x+\beta)}-\frac{\beta}{(\alpha+y)(a_n+y)}.$$

The first term here is greater than

$$\frac{\beta}{\beta+1},$$

and the other less than $$\frac{\beta}{\alpha a_n}.$$

The result $\lambda_n>\lambda$ follows at once.

4. As regards principal convergents, we have

$$\lambda_n=(a_{n+1}\,;\ a_{n+2},\ a_{n+3},\ \ldots)+(a_n,\ a_{n-1},\ a_{n-2},\ \ldots),$$

or say $$\lambda_n=\xi_n+\eta_n.$$

It is clear that $$\lambda_{n-1}=\xi_{n-1}+\eta_{n-1},$$

where $$\xi_{n-1} = a_n + 1/\xi_n,$$

$$1/\eta_n = a_n + \eta_{n-1},$$

so that $$\xi_{n-1} + \eta_{n-1} = 1/\xi_n + 1/\eta_n.$$

It follows that $$\lambda_n + \lambda_{n-1} = \xi_n + 1/\xi_n + \eta_n + 1/\eta_n;$$

and since $$\xi_n > a_{n+1}, \quad 1/\eta_n > a_n,$$

we deduce $$\lambda_n + \lambda_{n-1} > a_{n+1} + 1/a_{n+1} + a_n + 1/a_n.$$

In all cases therefore $$\lambda_n + \lambda_{n-1} > 4,$$

and, unless $$a_n = a_{n+1} = 1,$$

$$\lambda_n + \lambda_{n-1} > 4\tfrac{1}{2}.$$

It follows that except when, in the C.F. for θ, a_n is ultimately always unity, there is an infinite number of values of n for which $\lambda_n > 2\frac{1}{4}$.

In the exceptional case it is clear that, when $n \to \infty$.

$$\lambda_n \to \sqrt{5}.$$

To be more explicit, suppose a_r is the last a that is greater than unity, then

$$\lambda_n > \sqrt{5} \quad \text{for} \quad n-r \text{ odd},$$

and $$\lambda_n < \sqrt{5} \quad \text{for} \quad n-r \text{ even}.$$

Hence we have the result, first explicitly stated by Hurwitz, that for any irrational θ there is an infinite number of approximations x/y such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{y^2\sqrt{5}},$$

while, if $k > \sqrt{5}$, there are irrationals θ having only a finite number of approximations such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2}.$$

We have only to see, in fact, that, for all irrationals θ, $\lambda_n > \sqrt{5}$ for an infinite number of n's.

The class corresponding to a value of k is always infinite if $k \leqslant \sqrt{5}$, but if $k > \sqrt{5}$ it is finite for some θ's.

The simplest exceptional θ is

$$\theta = \tfrac{1}{2}(\sqrt{5}-1) = (1, 1, 1, \ldots).$$

5. Proceeding now to the proof of the Theorem (A), which is my main thesis, it may be remarked that, when an approximation x/y is assigned, there are only two fractions x'/y' ($x' < x$, $y' < y$) that satisfy the condition

$$xy' - x'y = \pm 1,$$

viz. one giving the upper and one the lower sign. Thus we have only to examine whether one of these two fractions is the member of the class that next precedes x/y.

If the condition for the class be too severe (*i.e.*, if the value of k be too large), the preceding member may be further away from x/y than either fraction x'/y', while if the condition be too lax the preceding member of the class may be nearer to x/y than either. This rough argument indicates the nature of the theorem and the manner of reasoning to be used in proof.

6. Another remark will be more directly useful. Suppose $x = p_n$, $y = q_n$, p_n/q_n being, as usual, the convergent to θ corresponding to a_n. Then, unless $a_n = 1$, the fractions x'/y' for which

$$xy' - x'y = \pm 1$$

are clearly the convergent p_{n-1}/q_{n-1} and the auxiliary one just preceding p_n/q_n.

If $a_n = 1$, the latter does not exist, and is replaced by the convergent p_{n-2}/q_{n-2}.

If x/y is the auxiliary convergent just before p_n/q_n, and $a_n > 2$, the two fractions x'/y' are p_{n-1}/q_{n-1} and the auxiliary convergent just before x/y.

If $a_n = 2$, the latter is absent and is replaced by p_{n-2}/q_{n-2}.

Finally, when x/y is the auxiliary convergent just after p_{n-1}/q_{n-1}, the two fractions in question are p_{n-1}/q_{n-1} and p_{n-2}/q_{n-2}.

These are all the cases that need be mentioned.

7. PROOF OF THEOREM (A).

I. If $k > 2$, all the approximations x/y must be principal convergents. We can readily construct a case in which, while p_{n+1}/q_{n+1} satisfies the condition, we have $a_{n+1} = 2$, but p_n/q_n does not satisfy it; for here

$$\lambda_n < 2 + \frac{1}{a_{n+2}} + \frac{1}{a_n},$$

and by taking a_n and a_{n-2} large enough we can ensure $\lambda_n < k$.

It follows (§ 6) that the member x/y next preceding p_{n+1}/q_{n+1} does not satisfy

$$p_{n+1}y - q_{n+1}x = \pm 1.$$

II. If $k = 2$, all the members are again principal convergents. If p_{n+1}/q_{n+1} satisfies the condition, then, if $a_{n+1} > 1$, p_n/q_n does also (2 . 1), while if $a_{n+1} = 1$, p_n/q_n may not. Yet since

$$\lambda_{n-1} + \lambda_n > 4,$$

either p_n/q_n or p_{n-1}/q_{n-1} must satisfy the condition; so that in either case (see § 6) the preceding member x/y does satisfy

$$p_{n+1}y - q_{n+1}x = \pm 1,$$

and the theorem is proved.

III. If $k < 1\frac{1}{2}$, we have only to indicate an exception to the relation

$$xy' - x'y = \pm 1.$$

We take $a_n = 3$, so that between p_{n-1}/q_{n-1} and p_n/q_n there are two auxiliary convergents, viz.,

p/q corresponding to $\alpha = 1,\ \beta = 2,$

p'/q' „ $\alpha = 2,\ \beta = 1;$

and we are going to arrange that p/q satisfies the class condition

$$\left| \frac{p}{q} - \theta \right| < \frac{1}{kq^2},$$

while p'/q' does not. In this case p_n/q_n must satisfy the condition (§ 3), and the preceding member p/q does not obey the formula

$$p_n q - q_n p = \pm 1.$$

A simple calculation is needed.

We can write

$$\lambda = \frac{1}{2+\xi} + \frac{1}{1+\eta},$$

$$\lambda' = \frac{1}{1+\xi} + \frac{1}{2+\eta},$$

$$\xi = (a_{n+1}, a_{n+2}, \ldots),$$

$$\eta = (a_{n-1}, a_{n-2}, \ldots),$$

and we have $$\lambda-\lambda' = \frac{1}{(1+\eta)(2+\eta)} - \frac{1}{(1+\xi)(2+\xi)},$$

so that $\lambda > \lambda'$ if only $\xi > \eta$.

It will be clear now that, k being less than $1\frac{1}{2}$ but ever so near it, we can so arrange a case that

$$\lambda > k \quad \text{and} \quad \lambda' < k;$$

this completes the argument for $k < 1\frac{1}{2}$.

What has been stated refers to cases where k is nearly $1\frac{1}{2}$, but to deal with other cases we need only take $a_{n+1} = 4$, 5, &c., instead of 3. The *crux* of the matter is that $\lambda > \lambda'$.

IV. The final case is $$1\tfrac{1}{2} \leqslant k < 2.$$

If an auxiliary convergent between p_{n-1}/q_{n-1} and p_n/q_n belongs to the class, we must have

$$a_n = 2, \quad \alpha = \beta = 1 \quad (\S\ 2).$$

Suppose now that p_n/q_n satisfies the condition

$$\left| \frac{p_n}{q_n} - \theta \right| < \frac{1}{kq_n^2}.$$

The preceding member of the class may be the intermediate convergent if $a_n = 2$, but if $a_n > 2$ it must be p_{n-1}/q_{n-1}; and both these satisfy

$$p_n x - q_n y = \pm 1 \quad (\S\ 6).$$

There remains the case $a_n = 1$. Clearly p_{n-1}/q_{n-1} may not belong to the class; if it does not, no auxiliary convergent between p_{n-2}/q_{n-2} and p_{n-1}/q_{n-1} can possibly belong (§ 3), while, since $\lambda_{n-2}+\lambda_{n-1} > 4$, p_{n-2}/q_{n-2} must belong.

This completes the proof as far as a principal convergent p_{n-1}/q_{n-1} is concerned.

It remains to discuss the member immediately before an auxiliary convergent that is like the one above. As indicated $a_n = 2$, $\lambda_{n-1} > 2$, and p_{n-1}/q_{n-1} is the preceding member.

Since it satisfies the condition, the proof is now finished.

8. In cases where successive approximations of the class satisfy the relation

$$xy' - x'y = \pm 1,$$

the whole class will differ from the class of principal convergents to θ in two possible ways :—

(i) if $a_{n+1} = 1$, the convergent p_n/q_n may have to be rejected ;

(ii) if $a_n = 2$, the auxiliary convergent between p_{n-1}/q_{n-1} and p_n/q_n may have to be inserted (§ 7, IV).

In these cases the C.F. for θ can be modified* (by the introduction of some minus signs, if necessary), so that the approximations of the class and no others occur as principal convergents.

Suppose, in fact, that x/y, x'/y', x''/y'' are three consecutive members of the class, so that

$$xy' - x'y = \pm 1, \quad x'y'' - x''y' = \pm 1.$$

We can write

$$x = \lambda x' + \mu x'',$$

$$y = \lambda y' + \mu y'',$$

and it readily follows that λ, μ are integers of which μ is ± 1 ; since λ must be positive this is the continued fraction algorithm. If $\mu = +1$, the partial quotient λ is preceded by a positive sign, otherwise by a negative sign.

A simple example will best indicate the method of modification.

Take $\theta = (4, 2, 4, 1, 4, \ldots)$:

then the principal convergents are

$$\tfrac{1}{4},\ \tfrac{2}{9},\ \tfrac{9}{40},\ \tfrac{11}{49},\ \tfrac{53}{236},\ \ldots,$$

and the auxiliary convergent between $\frac{1}{4}$ and $\frac{2}{9}$, viz. $\frac{1}{5}$, may belong, while the principal convergent $\frac{9}{40}$ may not. In fact, taking

$$\left| \frac{x}{y} - \theta \right| = \frac{1}{\lambda y^2},$$

we have for the first

$$\lambda > 1{\cdot}6,$$

* Cf. Hermite, *l.c.*, who deals with $k = \sqrt{3}$, and Minkowski (*Math. Ann.*, Vol. 54, p. 91 ; *Ges. Abhandlungen*, Vol. 1, p. 320), who deals with $k = 2$. The process is a definite one, as there stated, but not a direct one.

and for the second $\lambda < 1{\cdot}5$,

so that, with $k = 1{\cdot}6$, the first must be inserted and the second rejected.

First, change from $(4, 2, 4, 1, 4, \ldots)$,

to $(4, 2, 5, -5, \ldots)$:

the new convergents are $\frac{1}{4}, \frac{2}{9}, \frac{11}{49}, \frac{53}{236}, \ldots$,

and $\frac{9}{40}$ has disappeared.

Secondly, change to $(4, 1, 1, -6, -5, \ldots)$, and the convergents are

$$\frac{1}{4}, \frac{1}{5}, \frac{2}{9}, \frac{11}{49}, \frac{53}{236}, \ldots,$$

as desired.

The rules will be obvious and are easily established.

It may be mentioned that, since

$$\lambda_n + \lambda_{n+1} > 4,$$

two consecutive principal convergents have never to be rejected.

9. We have seen (§ 4) that, if

$$\theta = \tfrac{1}{2}(\sqrt{5}-1) = (1, 1, 1, \ldots),$$

and $k > \sqrt{5}$, there is only a finite number of rational approximations x/y to θ, such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2}.$$

A more general result has been established by Markoff,* viz. that, if θ has only a finite number of rational approximations such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{3y^2},$$

then θ must be a quadratic surd.

The first result, and some idea of Markoff's method, can be alternatively obtained by following the line of argument used by Liouville† in his classical proof of the existence of transcendent numbers.

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* *Math. Ann.*, Vol. 15, p. 381. It will, I think, be clear to any reader that the results of Hurwitz quoted earlier all really occur in this paper.

† *Liouville's Journal*, Vol. 16, p. 133. The argument has been often reproduced: see, *e.g.*, Young, *The Theory of Sets of Points*, p. 7.

In fact, with the above value of θ, we have

$$f(\theta) \equiv \theta^2+\theta-1 = 0,$$

whence

$$f\left(\frac{x}{y}\right)-f(\theta) = f'(\xi)\left(\frac{x}{y}-\theta\right),$$

where ξ is between x/y and θ. It follows that

$$\left|\frac{x}{y}-\theta\right| = \left|\frac{f(x/y)}{f'(x)}\right|.$$

Now, x and y being integers,

$$\left|f\left(\frac{x}{y}\right)\right| = \frac{\lambda}{y^2},$$

where λ is a positive integer, and so

$$f'(\xi) \to f'(\theta) = \sqrt{5},$$

when $x/y \to \infty$. We deduce that for any rational approximation

$$\left|\frac{x}{y}-\theta\right| \to \frac{\lambda}{y^2\sqrt{5}},$$

when $y \to \infty$; since $\lambda \geqslant 1$, the result follows at once, and it also follows that $\lambda = 1$ for the best approximations.

In general, when θ is a quadratic surd, λ will be the minimum of a certain indefinite quadratic binary form, and it was in finding λ that Markoff succeeded.

10. I shall now attempt to supplement the just quoted theorem of Markoff.

Suppose that, to make up the digits in the continued fraction for θ, we have a cycle of m 1's and another of m 2's, it being understood that each cycle can be used any number of times in succession. It is quite clear that the set of θ's so constructed is non-enumerable, and therefore that there are non-algebraic—*i.e.* transcendent—numbers among them; it is easy to see that the value of λ_n corresponding to a partial quotient a_n is distinctly less than 3, except when a_{n+1} is a 2 with a 1 and a 2 adjacent to it. In this case, when $m \to \infty$ and $n \to \infty$, $\lambda_n \to \frac{\sqrt{5}-1}{2}+\sqrt{2}+1$, say ϕ,

and it follows that, if $k > \phi$, there are transcendent numbers with only a finite number of approximations such that

$$\left| \frac{x}{y} - \theta \right| < \frac{1}{ky^2}.$$

The value of ϕ is $3{\cdot}0322 < 3\frac{1}{30}$.

It seems likely that, by a different choice of the two cycles, we could reduce the critical value of ϕ to 3, but I do not see at present how to do this. I may add that Markoff's result relating to the quadratic surds is by no means easy to establish; it probably does contain the key to the supplementary theorem here indicated.

TETRAHEDRA IN RELATION TO SPHERES AND QUADRICS

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Introduction.

This paper arose out of the following question :—"What are the relations between the inscribed and circumscribed spheres of a tetrahedron?" In other words, given two spheres S, S', what are the necessary and sufficient conditions that there should be *real* tetrahedra which are inscribed in S and circumscribed to S'?

It is proved (§§ 8, 9) that the sole condition is

$$(R+r)(R-3r) \geqslant d^2,$$

when R, r are the radii of the spheres and d is the distance between their centres; corresponding conditions for the escribed spheres are found, and in particular it is shewn that when two spheres are external to each other there are always real tetrahedra whose vertices lie on the one and whose faces touch the other.

The earlier sections are concerned with some general properties of quadrics which may be stated thus :—If the vertices of a tetrahedron lie on a quadric S and its faces touch a quadric S', then, when one face π is given, the opposite vertex lies on a certain plane section π' of S, and when π varies the envelope of π' is a quadric through the common curve of S and S'.

At the end of the paper will be found some simpler arguments that corroborate the general conclusions previously obtained; these simpler arguments are, in fact, more forcible in some ways than the earlier ones, but they are not so far-reaching.*

* I may add that the invariant of § 5 occurs in quite a different connexion in a paper by M. G. Fontené relating to certain polyhedra that have a hole through them, and are inscribed in the one quadric and circumscribed to the other. See *Nouv. Ann.*, 1904, p. 289. The quadric envelope of § 3 is alluded to by Kohn, *l.c. infra*, § 2. None of the papers quoted has reference to the reality of the tetrahedra. I cannot find that this question has been discussed before.

1. When a tetrahedron $LMNP$ has its vertices on a fixed quadric S, its position depends on eight arbitrary constants, so we are led to expect that the problem of finding a tetrahedron whose vertices lie on one quadric S and whose faces touch a second quadric S' will admit of ∞^4 solutions. This, as will presently appear, is usually the case; but, denoting the face LMN by π, we cannot choose the vertex P and the face π arbitrarily. In fact denote by σ and σ' the conics in which π cuts S and the tangent cone from P to S': then, if there were such a tetrahedron, there would be one triangle LMN whose corners lie on σ and whose sides touch σ'; hence there would be an infinite number of such triangles, and hence an infinite number of the tetrahedra for which the vertex P and the opposite face π are the same.

2. I proceed to shew that when the face π (it is of course a tangent plane to S') is given, the locus of the vertex P is in general a plane section of the quadric S. In addition to being on the quadric, P must be such that the conics σ and σ' satisfy the poristic condition already alluded to. Suppose, then, that a generator of S' in the plane π cuts S in H and K. Then, since HK is a tangent to the conic σ', the other tangents from H and K to σ' must meet on σ; but, if h and k are the other generators of S' through H and K, these two tangents are the lines in which the planes Ph, Pk cut the plane π, and it follows that these two planes must meet on the conic σ. Hence, the two pencils $h(P)$, $k(P)$ being homographic, P must lie on a quadric containing the lines h, k and the conic σ. As this quadric has the plane section σ in common with S, the remaining part of the intersection is also a plane section of S, and this is the required locus of the vertex P.

COR. I.—The quadric locus for P must also contain the other generators at the points H', K' in which S is cut by the other generator of S' in the plane π.

COR. II.—It might happen that the quadric locus for P coincided with S. In this case S and S' must have four common generators, and the quadric that contains them and the conic σ will coincide with S for all positions of π.

P and π can now be chosen arbitrarily, and there are ∞^5 tetrahedra inscribed in S and circumscribed to S'.* In other cases there are ∞^4,

* This theorem is due to M. G. Fontené, *Nouvelles Annales*, 1900, p. 67. It has also been discussed by Kohn, *Monatshefte für Math. und Phys.*, 1900, p. 102, and by Humbert, *Bull. Soc. Math. Fran.*, 1904, p. 135. The same writers naturally notice the fact that the locus of P, when π is given, is usually a plane section of S.

since when π is given there are ∞^1 positions for P, and corresponding to each such position there are ∞^1 tetrahedra.

3. I call the plane that contains P when π is given π', and proceed to prove that when π varies the envelope of π' is a quadric of the pencil

$$S+\lambda S'=0.$$

For this purpose I use elliptic parameters on the curve common to S and S'. They can be so chosen that if a, b, c, d are four coplanar points

$$a+b+c+d\equiv 0,$$

while if ab is a generator of a fixed quadric of the pencil

$$a+b=k,$$

a constant, and if cd is a generator of the opposite system

$$c+d=-k.$$

Now, referring back, let the generators h, k cut S again in α, β, and let the corresponding generators h', k', derived from $H'K'$ the other generator of S' in the plane π, cut S again in α', β'. Then we have

$$H+K=k, \quad H+\alpha=-k, \quad K+\beta=-k,$$

$$H'+K'=-k, \quad H'+\alpha'=k, \quad K'+\beta'=k,$$

and hence

$$\alpha+\beta=-3k,$$

$$\alpha'+\beta'=3k.$$

It follows that the lines $\alpha\beta$, $\alpha'\beta'$ are generators of a fixed quadric of the pencil, and so the plane $\alpha\beta\alpha'\beta'$, which is manifestly the plane π', always touches this fixed quadric.

4. Some further remarks will be useful.

(i) It is clear that, when π is given, the plane π' and its envelope depend only on the quadric S' and its curve of intersection with S—not on the quadric S in particular.

(ii) The plane π is touched by two quadrics of the pencil besides S', and the plane π' by two besides the envelope. These remaining tangent quadrics are the same for π and π'. In fact, if we write

$$H+H'=\lambda,$$

we have $\alpha+\alpha' = -\lambda, \quad \beta+\beta' = +\lambda, \quad K+K' = -\lambda,$

shewing that the lines HH', KK', $\alpha\alpha'$, $\beta\beta'$ are generators of the same quadric, and hence that π and π' touch this quadric. Similarly for the four lines

$$HK', \quad H'K, \quad \alpha\beta', \quad \alpha'\beta.$$

It follows, for example, that if π' touches S, and π does not touch S, then S is itself the envelope.

(iii) If the plane π is itself tangent to the envelope of π', the point of contact of π' is on the common curve of the pencil.

(iv) If the point of contact of π with S' is on the envelope of π', then π' is also tangent to S'.

5. I shall next find the equation of the envelope of π' when the equations of S and S' are

$$S \equiv ax^2+by^2+cz^2+dw^2 = 0,$$

$$S' \equiv x^2+y^2+z^2+w^2 = 0.$$

For this purpose take π to be

$$lx+my+nz+rw = 0.$$

The sections by π of S and the tangent cone from P to S' being σ and σ', we must have

$$\Theta'^2 = 4\Delta'\Theta,$$

the usual poristic condition. Eliminating w, a straightforward calculation shews that P must lie on the quadric whose equation is

$$\{\Sigma lx(b+c+d-a)\}^2$$
$$= 4\Sigma x^2\{l^2(cd+ab+bc)+(ac+ad)m^2+(ad+al)n^2+(ab+ac)r^2\}$$
$$+8\{admnyz+\ldots+bclrxw+\ldots\};$$

and thence that the equation of π' is

$$\alpha lx+\beta my+\gamma nz+\delta rw = 0,$$

wherein $\alpha = 8a^2-4a(a+b+c+d)+4(bc+ca+ab+ad+bd+cd)$
$$-(a+b+c+d)^2,$$

etc. Since $$l^2+m^2+n^2+r^2 = 0,$$

the envelope of π' is

$$\alpha^2x^2+\beta^2y^2+\gamma^2z^2+\delta^2w^2 = 0.$$

Of course we must have $$a^2 = Pa+Q,$$

where P, Q are symmetrical in a, b, c, d.

If we denote the discriminant of $S+\lambda S'$ by

$$\Delta+\lambda\Theta+\lambda^2\Phi+\lambda^3\Theta'+\lambda^4\Delta',$$

the zeros of this expression in λ are $-a$, $-b$, $-c$, $-d$; and Δ, Θ, &c., are now the fundamental invariants of the two quadrics.

An easy calculation now shews that the envelope is

$$PS+QS' = 0,$$

where $$P = 8\Delta'(\Theta'^3-4\Phi\Theta'\Delta'+8\Theta\Delta'^2),$$

and $$Q = (4\Phi\Delta'-\Theta'^2)^2-64\Delta\Delta'^3.$$

If we take S' to be a proper quadric, the condition that S itself should be the envelope of π' is $$(4\Phi\Delta'-\Theta'^2)^2 = 64\Delta\Delta'^3;$$

further the point of contact of π' with its envelope is always the pole of π for the quadric $$ax^2+\beta y^2+\gamma z^2+\delta w^2 = 0.$$

Hence, when S is the envelope of the planes π', we can choose each vertex to be the point of contact of the position of π' that corresponds to the opposite face, and all such tetrahedra are now self-conjugate for the quadric last mentioned, say F.

In verification we may note that

$$a+\beta+\gamma+\delta = 0,$$

so that there are ∞^3 tetrahedra inscribed in F that are self-conjugate for S', and, further, S and S' are reciprocals with respect to F. These results are analogous to known ones relating to triangles inscribed in one conic and circumscribed to another.

6. I now proceed to discuss the reality of the tetrahedra when the two quadrics are spheres, and begin by outlining the method.

(I) Suppose one of the faces π is fixed: then the opposite vertex will lie on a plane π'. It is necessary that π' should cut S in a real circle in some positions anyhow. Now the condition for a real section will be found by expressing the condition that a certain expression, which when equated to zero is the condition that π' should touch S, is positive. We know (§ 4) that π' only touches S either when S is the envelope or when π touches S.

so that the result will be largely independent of the position of π. In fact the expression arising will be the product of two factors, one of which will re-express the fact that π should meet S in a real curve, and the other the condition (§ 5) that S should be the envelope.

(II) It is not enough that the vertex P should be real: to give real tetrahedra it must be external to S'—it will be seen that a real section by π' is always external to S'.

(III) Even when the plane π is given and there are real vertices corresponding outside S', does it follow that some of the triangles inscribed in the circle σ and circumscribed to the conic σ' (both of which are real) are always real? The answer is "yes", though the argument is a little tedious.

7. It will be well to begin by calculating the invariant

$$(4\Phi\Delta'-\Theta'^2)^2-64\Delta\Delta'^3,$$

for two spheres
$$S \equiv (x-d)^2+y^2+z^2-R^2 = 0,$$
$$S' \equiv x^2+y^2+z^2-r^2 = 0,$$

whose radii are R, r, and whose centres are a distance d apart.

The simple invariants being

$$\Delta' = r^2, \quad \Theta' = R^2+3r^2-d^2, \quad \Phi = 3(R^2+r^2)-2d^2,$$
$$\Theta = 3R^2+r^2-d^2, \quad \Delta = R^2,$$

the above invariant is quickly seen to be

$$(R^2+2Rr-3r^2-d^2)(R^2-2Rr+r^2-d^2)$$
$$\times (R^2-2Rr-3r^2-d^2)(R^2+2Rr+r^2-d^2). \qquad (7.1)$$

8. Suppose now that the plane π is $z = 0$, and that the two spheres are

$$S \equiv x^2+y^2+z^2+2\alpha x+2\beta y+2\gamma z+\delta = 0,$$
$$S' \equiv x^2+y^2+z^2-2rz = 0,$$

so that
$$d^2 = (\gamma+r)^2+\alpha^2+\beta^2,$$
$$R^2 = \alpha^2+\beta^2+\gamma^2-\delta.$$

Proceeding directly, we find for the equation of π'

$$z\{\delta^2-4r^2(\alpha^2+\beta^2)\}-4r\alpha\delta x-4r\beta\delta y-4r\{\delta^2+2r^2(\delta-\alpha^2-\beta^2)+2r\gamma\delta\}=0.$$

We have now to set down the condition that this should cut S in a real circle, *i.e.* be within a distance R of the centre of this sphere. A calculation shews that the condition is

$$(R^2-\gamma^2)(R^2-2Rr-3r^2-d^2)(R^2+2Rr-3r^2-d^2)$$
$$(R^2-2Rr+r^2-d^2)(R^2+2Rr+r^2-d^2)\geqslant 0.$$

The first factor equated to zero is the condition that π should touch S, and the remainder is merely the invariant (7 . 1), so that all is as could have been predicted by § 6. If any of the other factors vanish, the sphere S is itself the envelope of the planes π'.

9. There will not be real tetrahedra unless

$$(R^2-2Rr-3r^2-d^2)(R^2-2Rr+r^2-d^2)(R^2+2Rr-3r^2-d^2)$$
$$(R^2+2Rr+r^2-d^2)\geqslant 0,$$

and of these factors the first is the least and the last the greatest.

Take, for clearness, the inequality first.

There are three cases—

(α) $$R^2-2Rr-3r^2-d^2>0,$$

or $$(R+r)(R-3r)-d^2>0,$$

or $$(R-r)^2>4r^2+d^2,$$

and all the other factors of course positive. Here the sphere S completely encloses the sphere S'.

(β) $$R^2+2Rr+r^2-d^2<0,$$

or $$R+r<d,$$

and all the other factors negative. Here the spheres are completely external to each other.

(γ) $$(R-r)^2<d^2,$$
$$(R+r)^2>4r^2+d^2.$$

In this case the spheres intersect in a real circle.

The case $(R-r)^2>d^2$, $(R+r)^2<4r^2+d^2$ is impossible if $R>r$,

and if $R < r$ it shews that S' completely encloses S. This is out of the question.

There are next the four cases in which the equality obtains in § 9, viz.

$$(\alpha') \quad (R-r)^2 = 4r^2+d^2,$$
$$(\beta') \quad (R+r)^2 = 4r^2+d^2,$$
$$(\gamma') \quad (R-r)^2 = d^2,$$
$$(\delta') \quad (R+r)^2 = d^2.$$

In each case π' touches S. In (γ') and (δ') S and S' touch, and it can be seen that π' touches them at their point of contact, so that these cases may be ignored.

(α') is merely the limiting case of (α) and (β') of (γ).

It may be noted that (β') implies $R < r+d$, so that the spheres cut in a real circle.

Only in cases (α), (β), (γ), (α'), (β') can there be real tetrahedra.

10. We have next to see if the vertices so found are external to S'.

In (α), (β), (α') the sphere S is completely external to S', so that any real vertex so found must be external to S'.

In the other two cases the spheres meet in real points, and the same result follows from a different argument.

The condition is $(R+r)^2 \geqslant 4r^2+d^2$.

Suppose S' is given, and Q is a given point common to S and S'. It is easy to see that the centre O of S must not be inside the paraboloid of revolution whose focus is Q, and whose vertex is O', the centre of S'; for, if OO' subtends an angle θ at Q, we have

$$R \cos^2 \tfrac{1}{2}\theta \geqslant r.$$

If now O' and the line of centres are given as well as Q, all the condition implies is that there is a point Ω on the line such that O may be anywhere not internal to the segment $\Omega O'$. Further O' and Ω are on opposite sides of the radical plane, and Ω is the centre of the envelope sphere. It is now easy to deduce that no tangent plane π' to the envelope sphere can meet S in points that are inside S'.

To sum up: in case any one of the following sets of conditions is satisfied, viz.,

$$(\alpha) \quad (R-r)^2 \geqslant 4r^2+d^2, \qquad (\beta) \quad R+r < d,$$
$$(\gamma) \quad (R+r)^2 \geqslant 4r^2+d^2, \quad R-r < d$$

and the plane π cuts S in real points, there are real positions of P on S that are external to S', such that the two conics σ and σ' are real and satisfy the poristic condition

$$\Theta'^2 = 4\Delta'\Theta \quad (\S\ 5).$$

11. I shall now shew that in such cases it is practically certain that there are real triangles whose sides touch σ', and whose vertices lie on σ. If P is any point on σ, the triangle with a vertex at P will be real if the tangents from P to σ' are real. Now, if the conics cut (but do not touch) in a real point A, there are points P on σ near A which are external to σ'; when the conics have two conjugate imaginary points of intersection a real projection will bring them into circles, and then the poristic condition shews either that σ encloses σ' or that σ and σ' meet in two real points. It follows that some of the triangles will be real except possibly when σ and σ' touch in two real points. This is an actual exception because the poristic condition then shews that at the common points the curvature of σ is four times that of σ', so that σ' encloses σ completely, and there are no real triangles. This sole exception is obviously a case of such rare occurrence that it might be ignored in a discussion like that of the tetrahedra, but I add one or two remarks.

From the reality of the conics σ and σ' follows the reality of the triangles (or some of them) except when σ and σ' have double contact. The condition of double contact is a double one of equality, and hence in general none of the positions of P found will satisfy it. A more minute argument confirms the conclusion, but, as the possible exception is a trivial one and the argument is only destined to prove a negative, I shall suppress it.

12. We have thus proved that the necessary and sufficient conditions that there should be real tetrahedra whose vertices lie on a sphere S and whose faces touch a sphere S', are

(α) S encloses S', and $(R-r)^2 \geqslant 4r^2+d^2$,

or (β) S and S' are external to each other,

or (γ) S and S' meet in a real circle, and $(R+r)^2 \geqslant 4r^2+d^2$.

In general there are ∞^4 of the tetrahedra, but in case of the sign

of equality there are ∞^3, and then (§ 5) the tetrahedra are self-conjugate with respect to a fixed quadric.

13. There are three types of spheres that touch the faces of a tetrahedron, viz. :—

(α) the inscribed sphere ;

(β) the doubly escribed spheres, which touch two of the faces on the opposite side to the inscribed sphere, and the other two on the same side ;

(γ) the ordinary escribed spheres, which touch one face on the opposite side to the inscribed sphere, and the other three faces on the same side.

It seems natural to suppose that this classification coincides with that in § 12. To establish this we have only to prove that an escribed sphere (γ) always cuts the circum-sphere in a real point, while an escribed sphere (β) is always external to the circum-sphere—it is obvious that the inscribed sphere is enclosed by the circum-sphere. The proofs required are simple enough if we assume a known theorem,* to the effect that the spheres touching the faces of a tetrahedron $ABCD$ can be divided into four pairs, such that the centres of a pair are collinear with D, and that a pair are touched by the same sphere through the circle ABC. There will be no need to give the argument in detail—it is simple when the theorem alluded to is assumed and that theorem is not quite simple.

14. It may be of some interest to give an argument relating to the inscribed sphere, which does not depend on the, sometimes elaborate, apparatus used before.

We have two spheres S and S', of which the first encloses the second, and we want the necessary and sufficient condition that there should be real tetrahedra inscribed in S and circumscribed to S'.

The case in which the spheres are concentric is particularly simple. Any real tetrahedron satisfying the conditions must now have its opposite edges equal in pairs. For if it be $ABCD$, the circumcentre of the face ABC, being the point of contact of the inscribed sphere with that face, must be internal to the face, and the triangle ABC must thus be acute-angled.

* *Camb. Phil. Soc. Trans.*, Vol. 16, p. 166.

Again, the circles ABC, ABD being equal, the angles ACB, ADB are equal or supplementary; as both are acute they must be equal.

It follows that the angles at any corner, say D, are equal to the angles in the opposite face ABC, and thence we deduce that the tetrahedron must be isosceles, as stated.

15. Conversely, if a tetrahedron is isosceles, the faces are acute-angled and the circum-centre and in-centre both coincide with the centroid. Also the radius of the inscribed sphere is one-fourth of each perpendicular of the tetrahedron, and hence at most one-third of the radius of the circumscribing sphere.

16. Take now two concentric spheres S and S' of radii R and r ($R > r$). A plane π touching S' at H will cut S in a circle σ whose centre is H, and the tangent cone from any point R on S to S' in a conic σ', of which H is a focus. If z is the distance of R from π and t the tangent from R to S', the major axis $2a$ of σ' is given by

$$2a = 2t\frac{r}{z-2r}.$$

When this major axis is equal to the radius of the circle σ, which is t, there will be triangles inscribed in σ whose sides touch σ' (§ 2). Thus there will be tetrahedra with one vertex at R, and the other three in the plane π, which are inscribed in S and circumscribed to S'. The condition for this is

$$z = 4r;$$

so R must lie on a plane parallel to π and at a distance $4r$ from it. This plane cuts S in a real circle only if $3r < R$: it touches S if $3r = R$.

All this corroborates §§ 8, 9, and shews that, if $3r \leqslant R$, there are real tetrahedra having S for circumscribing sphere and S' for inscribed sphere. There are ∞^4 if $3r < R$ and ∞^3 if $3r = R$. The tetrahedra are all isosceles (of course regular if $3r = R$) and their altitudes are $4r$.

17. Passing to the general case in which S and S' are not concentric, I first shew that if there is one real tetrahedron inscribed in S and circumscribed to S', then there exists such a tetrahedron, one of whose faces is an arbitrary tangent plane π of S'.

Suppose $LMNP$ is the given tetrahedron: the plane LMN will cut S and the tangent cone from P to S' in two conics σ and σ', of which σ encloses σ' completely. Hence there will be a tetrahedron with one corner at P and another anywhere on the circle σ. The vertices L, M,

N, P cannot all be on the same side of the plane π; if L and M, say, are on opposite sides, it is clear that the sections of S by the planes LMN and π have two real points in common. It follows that we can bring one vertex, say L, of the tetrahedron into the plane π without moving P.

Repeating the process and keeping L fixed, we can bring a second vertex, say M, into the plane π. The two remaining vertices are one on each of the tangent planes through the line LM to S'; thus one of them is on π, and π is now a face.

The argument would apply equally well to two ellipsoids: it depends merely on Poncelet's Porism, and the fact that one surface is entirely inside the given tetrahedron, while the other is all but completely external to it.

18. We are thus justified in finding the necessary and sufficient condition for the existence of real tetrahedra by taking one face to be any tangent plane to the inscribed sphere S'. Naturally one of those whose point of contact is on the line joining the centres is chosen.

Call this plane ϖ and the opposite vertex P: the radii of the spheres are R, r, and the distance between the centres is d: z is the distance of P from ϖ, and t the tangent from P to S'. Then the major axis $2a$ of the conic σ', in which ϖ cuts the tangent cone from P to S', is given by

$$a = t\,\frac{r}{z-2r}. \qquad \text{(II)}$$

The section of S by ϖ is a circle σ of radius ρ, say, whose centre is the point O when ϖ touches S', and the poristic relation between σ and σ' is

$$\rho = 2a.$$

Now, using O for origin, the equations of the spheres will be

$$S \equiv x^2+y^2+z^2-2cz-\rho^2 = 0,$$

$$S' \equiv x^2+y^2+z^2-2rz \quad = 0.$$

The point P must satisfy the conditions

$$\rho^2(z-2r)^2 = 4r^2(x^2+y^2+z^2-2rz),$$

from (II), and $\qquad x^2+y^2+z^2-2cz-\rho^2 = 0.$

We thus get an equation for z which reduces to

$$\rho^2 z = 4\rho^2 r+8r^2c-8r^3, \qquad \text{(III)}$$

and the distance of P from $z=0$ must not be greater than $c+R$. Thus we have

$$\rho^2(R+c) \geqslant 4\rho^2 r+8r^2c-8r^3,$$

and, of course,

$$R^2=\rho^2+c^2.$$

We deduce

$$(R^2-c^2)(R+c) \geqslant 4(R^2-c^2)r+8r^2c-8r^3;$$

and, since

$$d+r=c,$$

this gives

$$\{R^2-(r+d)^2\}\{R+d-3r\} \geqslant 8r^2d,$$

or

$$R^3-R^2(3r-d)-R(r+d)^2+(r-d)(3r^2+d^2) \geqslant 0,$$

or

$$(R-r+d)(R^2-2Rr-3r^2-d^2) \geqslant 0.$$

Clearly S' must be inside S, so that the first factor is positive, and we are left with

$$(R+r)(R-3r) \geqslant d^2,$$

or

$$(R-r)^2 \geqslant 4r^2+d^2 \quad \text{(cf. §§ 8, 9)}.$$

ON THE CURVES WHICH LIE ON THE QUARTIC SURFACE IN SPACE OF FOUR DIMENSIONS, AND THE CORRESPONDING CURVES ON THE CUBIC SURFACE AND THE QUARTIC WITH A DOUBLE CONIC

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THE quartic surface in space of three dimensions which has a double conic was discussed by Segre (*Math. Annalen*, Vol. 24, 1884) as the projection, from an arbitrary point, of the quartic surface Γ in space of four dimensions which is the intersection of two quadratic varieties in that space. When the centre of projection is taken on Γ itself, the surface projects into a cubic surface S_3. Part I of this paper begins with a short account of some elementary properties relating to four-dimensional space and a brief résumé of the chief properties of Γ proved by Segre. I then proceed to show how the properties of lines, conics, and cubics on S_3 can be deduced from the properties of the corresponding curves on Γ. But it is found that the cubics on Γ do not exhaust the cubics on S_3. In Part II, I therefore develop a theory of coresiduation on Γ. This part of the paper follows closely a paper by Prof. Baker (*Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1912). A good deal of this work could have been deduced as a generalisation of the work in Prof. Baker's paper, but a development from first principles was thought preferable. It is shown that all curves on Γ can be expressed in terms of six fundamental elements. The concept of a double six on Γ is then introduced, and it is shown that there exist conical varieties in space of four dimensions with properties similar to those of Schur's quadrics, these being quadrics associated in a particular manner with the cubic curves on a cubic surface.* In Part III, the results obtained in Part II are projected, and a corresponding theory of coresiduation on the quartic surface in three dimensions which possesses a double conic is briefly sketched. It

* See Baker, *loc. cit.*, p. 300, and Reye, referred to below.

is known* that the lines on this surface can be grouped in double fours: and it is here found that the double four is connected with rational quartics on the surface in the same way as cubics on S_3 are connected with the double six. The work is in the first place an example of co-residuation, and in the second place an illustration of the principle of double projection which consists in a projective generalisation from the cubic surface to Γ, followed by a projective particularisation from Γ to S_4, the quartic with a double conic.

Throughout the preparation of this paper I have been in constant consultation with Prof. Baker. I render to him my grateful thanks.

The following is a list of the more important references which I found useful in the preparation of the paper:—

1. Segre, *Math. Annalen*, Vol. 24, 1884, pp. 313–343.

2. Jessop, *Quartic Surfaces*, 1916, Chapter III.

3. Baker, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1912, pp. 285–301.

4. Reye, *Math. Annalen*, Vol. 55, 1902, pp. 257–264.

5. Berzolari, *Annali di Math.*, Ser. 2A, Vol. 13, 1885, pp. 102–120.

PART I.

1. Any point in space of four dimensions may be represented by five homogeneous coordinates x, y, z, t, u. A locus of three dimensions represented by one rational equation is called a variety or threefold. An ∞^2 locus is called a surface or twofold, and is given by two equations. An ∞^1 locus is called a curve or onefold, and is determined by three equations. When these loci are linear, they are called hyperplane, plane, and line respectively. The general hyperplane is represented by $\Sigma ax = 0$: a plane by two such equations, a line by three, and a point by four such equations. Two planes intersect in a line or a point only, according as they do or do not lie in the same hyperplane. A hyperplane is determined by four arbitrary points and a plane by three.

An algebraic variety met by any line in two points is termed a quadratic variety or simply a quadratic. Its equation is of the form

$$F_2(x, y, z, t, u) = 0,$$

* See Salmon, *Solid Geometry*, Vol. 2, 1915, p. 222.

where F_2 is a homogeneous quadratic function of the five variables. Its section by any hyperplane is a quadric surface, called here simply a quadric, and to be distinguished from a quadratic; and its section by any plane is a conic. The general equation of a quadratic involves fourteen constants. But just as a quadric may degenerate into a cone or a pair of planes, so also a quadratic may degenerate into a cone of the first species, a cone of the second species, or a pair of hyperplanes, according as the equation is reducible by linear substitutions to a form involving four, three, or two variables only. The equation of the general quadratic may be brought to the form $\Sigma x^2 = 0$ by suitable choice of the coordinate hyperplanes. The equations of the three degenerate forms may similarly be brought to the forms

$$x^2+y^2+z^2+t^2 = 0, \quad x^2+y^2+z^2 = 0, \quad x^2+y^2 = 0.$$

It follows from the first of the above equations that there is one point intimately associated with a cone of the first species or hypercone as it may be called. Whereas an arbitrary hyperplane intersects the hypercone in a quadric, any hyperplane passing through this particular point meets it in a quadric cone with vertex at this point. For this reason this point is called the vertex of the hypercone. From this it follows that if we join every point of a quadric to a point O which does not lie in the same hyperplane, we obtain a hypercone with vertex at O. Every tangent hyperplane to the hypercone passes through the vertex, and intersects the surface in two planes whose line of intersection is the line of contact of the tangent hyperplane. These planes are called generator planes, since they correspond to the generators of a quadric. Also they fall into two classes such that two planes of the same class have only one point in common, viz., the vertex of the hypercone, whereas two planes of opposite classes have a line common. Thus a hyperplane can always be drawn through two generator planes, one of each class.

The second degenerate form of the quadratic is the cone of the second species. Its equation being of the form

$$x^2+y^2+z^2 = 0,$$

every tangent hyperplane passes through a fixed line which is called the vertex line. The section by any hyperplane is a quadric cone with its vertex on the vertex line.

The principles of reciprocation apply equally well to the case of quadratics. Every point has a polar hyperplane, and every line has a polar plane. In the case of a hypercone the polar hyperplane of any point passes through the vertex.

2. We consider now the intersection of two quadratic varieties. Two quadratics whose discriminant does not vanish* can simultaneously be reduced to the forms

$$F = \Sigma x^2 = 0, \quad \Phi = \Sigma a x^2 = 0.$$

Their intersection is a quartic surface or twofold. It is this surface which Segre projects from an arbitrary point on to an arbitrary hyperplane, so obtaining a quartic surface in space of three dimensions possessing a double conic. We may call it Γ with Segre. The network of quadratics $F+\lambda\Phi = 0$ contains five hypercones, one of which is

$$(a-e)x^2+(b-e)y^2+(c-e)z^2+(d-e)t^2 = 0.$$

The five cones give rise to ten sets of generator planes. We shall now show that each of these planes meets Γ in a conic. Any hyperplane meets Γ in a quadriquartic curve; in particular, a tangent hyperplane to any one of the hypercones. Γ lies entirely on each of the hypercones; and we have seen that a tangent hyperplane to a hypercone meets it in two generator planes. Hence the quartic curve of intersection of Γ by any such tangent hyperplane must break up into two plane curves lying in the two generator planes and intersecting twice on the line common to the two planes. These plane curves must obviously be conics. Thus any generator plane meets Γ in a conic. Conversely, every conic on Γ must lie in one of the generator planes.

There are further sixteen lines on Γ, falling into groups in a special manner with respect to each hypercone. The analytical proof is extremely simple, and reference may be made to Prof. Jessop's treatise.† The argument given by Segre himself is interesting as going to the root of the matter, and may be exhibited as follows. The vertices of the five hypercones belonging to the network $F+\lambda\Phi = 0$ form a pentahedron which is self-polar with respect to all the quadratics of the network. Consider one of the vertices and the hypercone through that point. The opposite hyperplane intersects this hypercone in a principal quadric and Γ in a quartic curve. In each of the two systems of generators of the principal quadric there are four which touch the quartic curve. Let g be one of these generators, and P its point of contact with the quartic curve. Then the plane joining g to the vertex is a generator plane of the hypercone we started with. Further this plane touches Γ at P; for the tangent hyperplane at P to any quadratic of the network must pass through the

* Bertini, *Intro. Geo. d. Iperspazi* (Pisa, 1907), Cap. 7, No. 7.

† *Loc. cit.*, p. 57.

vertex because P lies on the polar hyperplane of the vertex, and it must also contain g because g is a tangent to a particular curve on Γ. We shall now show that this tangent plane to Γ meets Γ in two lines. Any quadratic of the network is met by its tangent hyperplane at P in a quadric cone with P for vertex. This quadric cone is met by the tangent plane at P to Γ in two lines through P. Thus the generator plane we are considering meets Γ in two lines through P.

Corresponding to the four generators of the one system we thus obtain four pairs of intersecting lines on Γ, which we may call b_6a_1, $c_{23}c_{45}$, $c_{24}c_{53}$, $c_{25}c_{34}$. None of these four pairs intersects any other, because the four pairs are situated in four generator planes of the same system. So also the four generators of the other system of the principal quadric lead to four pairs of intersecting lines on Γ, which may be called a_2c_{21}, a_3c_{31}, a_4c_{41}, a_5c_{51}. None of these pairs intersect among themselves. But we have seen that two conics situated in generator planes of opposed systems intersect twice. Hence any one of the one group of four pairs meets each of the four pairs of the other group in two points. We name the latter group of lines in such manner that a_2 meets $b_6c_{23}c_{24}c_{25}$; and c_{21} meets $a_1c_{45}c_{53}c_{34}$; and so on. We thus obtain sixteen lines on Γ arranged in two groups of four pairs such that each line has five secants. We may therefore represent the sixteen lines by the following scheme, following Schläfli's notation :—

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & . \\ . & . & . & . & . & b_6 \\ c_{12} & c_{13} & c_{14} & c_{15} & . \\ c_{23} & c_{24} & c_{25} & . \\ c_{34} & c_{35} & . \\ c_{45} & . \\ . \end{array}$$

The sixteen lines lie on each of the five hypercones; the grouping into two groups of four pairs each is different with reference to the five hypercones, as will be shown later on.

3. We now consider the result of projecting Γ from an arbitrary point O on itself to an arbitrary hyperplane R_3. The projection is a cubic surface S_3. The sixteen lines on Γ project into sixteen lines on S_3. Whence arise the other eleven lines on the cubic? We will show that they arise as the projections of eleven conics on Γ. Through an arbitrary point O on Γ there pass two generator planes of each of the five hypercones, one

plane of each system. This is so because through any point of a quadric there pass two generators, one of each system. The ten generator planes thus obtained meet Γ in ten conics, each of which passes through O. These ten conics may be called the fundamental conics on Γ, and they project into lines on S_3. Further the tangent hyperplanes at O to the quadratics of the network all pass through a particular plane ϖ which is the tangent plane at O to Γ. Any line through O lies in one or other of these hyperplanes. But a line through O in the plane ϖ intersects Γ in two points at O. In other words, the plane ϖ is the locus of all the lines through O that meet Γ in two points at O. These points of Γ may be said to lie on a zero circle with O as centre lying in the plane ϖ. The plane ϖ meets R_3 in a line which lies on S_3. There are thus twenty-seven lines on S_3; and it is easy to see that there cannot be any more. For a line on S_3 can only result from a plane curve on Γ, and an arbitrary line meets F and Φ, the two fundamental quadratics, in two points each; hence it cannot meet Γ in more than two points. Thus a plane curve on Γ must be either a conic or a line, and we have considered these cases. Thus there cannot be a twenty-eighth line on S_3.

4. We proceed to discuss the relations of the twenty-seven lines as regards mutual intersections. The twenty-seven lines fall into two groups, one of sixteen due to the lines on Γ, and the other of eleven due to the conics on Γ. We have seen in § 2 how the sixteen lines on Γ intersect among themselves. These relations remain on projection. It is enough, therefore, to discuss the mutual intersections of the group of eleven lines among themselves, and the intersections of the lines of this group of eleven with the lines of the group of sixteen. We shall prove that—

(A) the line due to the zero circle meets each of the lines due to the ten conics;

(B) the line due to any one of the ten conics is met, besides, by the line due to one other of the conics;

(C) the line due to the zero circle does not meet any of the sixteen lines;

(D) the line due to any one of the ten conics meets eight of the sixteen lines.

The truth of (A) follows from the fact that ϖ lies in a hyperplane with each of the ten generator planes through O, the hyperplane being the tangent hyperplane at O to the hypercone to which the generator plane be-

longs. (B) follows from the fact that the ten conics can be paired, as we have seen, into five couples such that each couple lies in a hyperplane through O. (C) is obvious, for O is chosen arbitrarily after Γ is given; that is, the zero circle presents itself after the position of the sixteen lines is fixed. To prove (D) consider any one conic; the plane of this conic is a generator plane of one of the hypercones. The opposite system of generator planes of this hypercone contains four planes, each of which meets Γ in a pair of lines. It follows that the plane of the conic we are considering lies in a hyperplane with each of the four generator planes mentioned. Hence the line due to this conic meets eight of the sixteen lines. It will not meet the other eight, because these are situated in generator planes of the same system as that to which the conic belongs. Calling the line due to the zero circle a_6, and naming the ten conics in accordance with the intersections we have found, we can complete the incomplete Schläfli scheme at the end of § 2. The ten conics are b_r and c_{r6}, where r varies from 1 to 5.

5. We now proceed to consider conics on S_3. These being plane curves can arise only as the projections of hyperplanar curves on Γ, the hyperplanes passing through O. Such curves are conics not passing through O, cubics through O, and quartics passing twice through O. Any conic on Γ lies in a generator plane of one of the hypercones. This plane lies in a hyperplane together with that generator plane of the opposed system of the same hypercone that passes through O. Hence the corresponding conic on S_3 lies in a plane with one of the lines due to the ten conics. Any cubic on Γ must lie in a hyperplane with one of the sixteen lines on Γ. Lastly, any hyperplane through ϖ meets Γ in a quartic passing twice through O. This quartic projects into a conic on S_3 lying in a plane with a_6. We thus obtain the twenty-seven systems of conics on S_3 lying on the planes through the twenty-seven lines.

It is now easy to establish the theorem that three conics on S_3 lying in planes through three lines that form a triangle lie on a quadric. This is not difficult in itself, but it leads up to a point worth consideration. A triangle of three lines on S_3 can arise in only two ways. (A) The zero circle and the two fundamental conics of any one hypercone lead to a triangle of lines. There are five triangles of this type. (B) Any one of the ten conics together with any one of the four pairs of lines that belong to the opposed system of generator planes of the same hypercone lead to a triangle; and there are forty of this type. Consider the first case. The residual intersections of three central hyperplanes through the three elements mentioned are a quartic with a double point at O and two conics.

Since these conics do not pass through O, and since the quartic is a hyperplanar curve, each of the conics meets the quartic in two points; also the two conics meet in two points, for the two fundamental conics of which they are the residues meet in two points. In the second case the residual intersections are a conic not passing through O, and two cubics through O. The lines of which these cubics are the residues meet in one point; hence the cubics meet in three points of which O is one. Further the residual conic meets each of the cubics in two points, for it does not meet either of the associated lines. Thus in both cases, on projection, we obtain a set of three conics of which each two meet in two points; whence they lie on a quadric. Moreover this quadric is only the section by R_3 of a hypercone with vertex at O, on which the three residual elements in (A) or in (B) are situated.

We shall prove that a unique hypercone with vertex at O can be drawn to pass through the three residual elements in each case. Consider Case (A). Besides the six points of mutual intersection, take one more point on each. Through these nine points a unique hypercone can be drawn with vertex at O. The hypercone will contain both the conics, for it passes through five points of each. It will also pass through the quartic, for the quartic projects from O to any hyperplane into a conic, and the quadric in which the hyperplane meets the hypercone passes through the conic, since it goes through five of its points. An identical argument holds in Case (B). Thus in each case a unique hypercone with vertex at O can be drawn to pass through the three residual elements. It appears from this that, whenever a quadric is associated geometrically with a set of curves on S_3, there is a hypercone with vertex at O which is similarly associated with the corresponding curves on Γ.

6. We pass on to consider cubics on Γ. A cubic curve in space of four dimensions is at the most a hyperplanar curve. Hence we can obtain cubics on Γ only by passing hyperplanes through any of the lines on Γ. Thus any cubic on Γ is associated with one of the lines on Γ, which it meets in two points, since together they constitute a quartic of the first species. Hence any two cubics associated with the same line meet in one point only. Further, denoting any one of the sixteen lines by l, the five lines that meet it by m and the ten lines that do not meet it by n, we shall call any cubic associated with l an L cubic, any cubic associated with one of the m lines an M cubic, and similarly for N cubics. Since l meets m, it follows that an L cubic meets an M cubic in three points. But l does not meet n; hence l meets an N cubic, and n meets an L cubic, in one point, whereas an L cubic meets an N cubic in two points.

We have so far found only sixteen double infinities of twisted cubics on S_3, for a double infinity of hyperplanes pass through a given line. Obviously, then, we have not been able to exhaust the cubics on S_3, for it is known that there are seventy-two infinities of them. A hyperplanar quartic on Γ can only give rise to a quartic or a plane cubic on S_3, according as the hyperplane does not or does pass through O. The other curves on Γ that give rise to cubics on S_3 must, therefore, be non-hyperplanar curves with a multiple point at O. We now pass on to a study of co-residuation on Γ.

PART II.

7. We have said before that the grouping of the sixteen lines into eight line pairs is different for each of the five hypercones. The following table indicates the grouping into line pairs with reference to each of the hypercones. It also indicates, with reference to each hypercone, the division of the eight pairs into two groups of four each, such that the two groups belong to the two systems of generator planes. Every pair in brackets such as $(a_2 c_{21})$ is a pair of lines which together constitute a degenerate conic on Γ. It is seen that any one line enters into five such brackets, once with each of its five secants. We have already seen that two conics situated in generator planes of opposite systems meet in two points. But two generator planes of the same system meet only in the vertex of the hypercone. Hence the conics in these planes do not meet at all. As an example, consider the line pair $(a_2 c_{21})$. This conic meets twice each of the conics $(b_6 a_1)$, $(c_{23} c_{45})$, $(c_{24} c_{53})$, $(c_{25} c_{34})$, because these belong to the opposite system; and it does not meet any of the conics $(a_3 c_{31})$, $(a_4 c_{41})$, $(a_5 c_{51})$, because these belong to the same system of generator planes as $(a_2 c_{21})$ itself. All these statements can be verified with the aid of the Schläfli scheme. The distribution of the ten fundamental conics is also shown, these being placed in square brackets. We shall use the word cone instead of hypercone when there is no danger of confusion.

Cone 1.		Cone 2.		Cone 3.		Cone 4.		Cone 5.	
$(a_2 c_{21})$	$(b_6 a_1)$	$(a_3 c_{32})$	$(b_6 a_2)$	$(a_4 c_{43})$	$(b_5 a_3)$	$(a_5 c_{54})$	$(b_6 a_4)$	$(a_1 c_{15})$	$(b_6 a_5)$
$(a_3 c_{31})$	$(c_{23} c_{45})$	$(a_4 c_{42})$	$(c_{34} c_{51})$	$(a_5 c_{53})$	$(c_{45} c_{12})$	$(a_1 c_{14})$	$(c_{56} c_{23})$	$(a_2 c_{25})$	$(c_{12} c_{34})$
$(a_4 c_{41})$	$(c_{24} c_{53})$	$(a_5 c_{52})$	$(c_{35} c_{14})$	$(a_1 c_{13})$	$(c_{41} c_{25})$	$(a_2 c_{24})$	$(c_{52} c_{31})$	$(a_3 c_{35})$	$(c_{13} c_{42})$
$(a_5 c_{51})$	$(c_{25} c_{34})$	$(a_1 c_{12})$	$(c_{31} c_{45})$	$(a_2 c_{23})$	$(c_{42} c_{51})$	$(a_5 c_{34})$	$(c_{53} c_{12})$	$(a_4 c_{45})$	$(c_{14} c_{23})$
$[c_{61}]$	$[b_1]$	$[c_{62}]$	$[b_2]$	$[c_{63}]$	$[b_3]$	$[c_{64}]$	$[b_4]$	$[c_{65}]$	$[b_5]$

8. We now proceed to state and prove a lemma regarding the number of points through which a unique curve of order $4r$ on Γ can be drawn. The lemma is as follows.

The effective number of constants in a variety V_r of order r, so far as its curve of intersection with Γ is concerned, is $2r(r+1)$.*

To prove the lemma, we observe that when the discriminant does not vanish F and Φ can simultaneously be reduced to the forms

$$F = -t^2+x^2+y^2+z^2 = 0,$$

and

$$\Phi = -u^2+ax^2+by^2+cz^2 = 0.$$

The general quadratic V_2 contains fourteen constants. There are two terms in V_2, one of which contains t^2 and the other u^2. Each of these can be replaced in terms of x, y, z by means of $F = 0$ and $\Phi = 0$. Hence only twelve constants are left. Similarly V_r contains

$$(r+4)(r+3)(r+2)(r+1)/24$$

terms. Of these, the number of terms divisible by t^2 is

$$(r+2)(r+1)\,r\,(r-1)/24,$$

this being the number of terms in $(x, y, z, t, u)_{r-2}$. In each of these we can replace the highest power of t^2 that occurs by a power of $x^2+y^2+z^2$ with the aid of the equation $F = 0$. Similarly the number of terms in V_r divisible by u^2 is $(r+2)(r+1)\,r\,(r-1)/24$. In each of these again we can replace the highest power of u^2 that occurs by an appropriate power of $ax^2+by^2+cz^2$ with the aid of the equation $\Phi = 0$. But in this enumeration we have counted twice the terms that are divisible by u^2t^2; and these are $r(r-1)(r-2)(r-3)/24$ in number. Hence the number of absolute constants left in V_r is

$$\tfrac{1}{24}(r+4)(r+3)(r+2)(r+1)-1-\tfrac{2}{24}(r+2)(r+1)\,r\,(r-1)$$
$$+\tfrac{1}{24}r(r-1)(r-2)(r-3) = 2r(r+1).$$

The lemma is therefore proved.

9. We now deduce the following theorem :—

Given any curve of order m on Γ, we can draw a variety of order r to intersect Γ in the given curve, together with a curve or set of curves each of lower order than the original.

* This result is due to Prof. Baker.

It follows from the lemma that a variety V_r can be made to contain the given curve of order m entirely, if r is chosen so that $2r(r+1)$ is greater than rm; and the residual curve of intersection with Γ will be of order $4r-m$. Now when $m=2k+1$, we can take $r=k$, and the residual curve is of order $4r-m=m-2$. When $m=2k$, we may take $r=k$, and the residual curve is of order m. In the latter case we could compel the variety to go through a straight line on Γ, besides going through the given curve of order m; this would exhaust $r+1$ of the available constants in V_r. It is obvious that the relations $m=2k$ and $r=k$ allow the inequality

$$2r(r+1)-(r+1)>rm,$$

for this reduces to $k>1$. Hence the residue in the latter case consists of a line and a curve of order $m-1$. The theorem is thus proved.

10. Two curves A and B on Γ are said to be coresidual, if there exists a curve or set of curves C on Γ, such that one variety may be drawn to meet Γ in A and C, and another of the same order to meet Γ in B and C. The fact that A and B are coresidual may be represented by the symbol of equivalence $A\equiv B$. Since any two hyperplanes together constitute a quadratic variety, it follows that the sections of Γ by any two hyperplanes are coresidual.

Consider, now, some particular cases of the theorem proved in the last section. Any cubic is residual with a line, and hence coresidual with a line and a conic. Any quartic is residual with a line and a cubic, and hence coresidual with two lines and a conic. Any quintic is residual with a cubic, and hence coresidual with a line and two conics. More generally, it follows from the theorem that any curve on Γ is coresidual with a curve which consists of a certain number of lines and a certain number of conics. Thus any curve on Γ may be expressed symbolically in terms of lines and conics.

Passing on to conics, we shall prove the following two results. First, if we fix the ten conics through any one point, for example, the ten fundamental conics through O, any other conic on Γ is coresidual with one or other of these. For instance, any conic which belongs to the same system of generator planes as c_{h6} is coresidual with c_{h6} itself, for each of these two conics is residual with b_h. So also any conic which belongs to the same system of generator planes as b_h is coresidual with b_h. Secondly, each of the ten fundamental conics is coresidual with a pair of lines. This is obvious, for c_{h6} is coresidual with $c_{hk}+a_k$ and b_h is coresidual with b_6+a_h. Thus any conic on Γ can be expressed in terms of lines; and we

have seen above that any curve on Γ is expressible in terms of lines and conics. It follows, therefore, that any curve on Γ may be expressed symbolically as the coresidual of a curve which consists entirely of lines, each taken with an appropriate order of multiplicity.

11. It will now be shown that we can choose six of the lines in such manner that the remaining ten are expressible in terms of these. Suppose we select a_1, a_2, a_3, a_4, a_5 and c_{12}. From Cone 1 we observe that a_3+c_{13} is residual with b_6+a_1; so also is a_2+c_{12}. Hence a_3+c_{13} is coresidual with a_2+c_{12}. This we write as

$$a_3+c_{13} \equiv a_2+c_{12},$$

and we agree to understand the same thing by

$$c_{13} \equiv c_{12}+a_2-a_3. \tag{1}$$

Similarly, we obtain from Cone 1 the further relations

$$c_{14} \equiv c_{12}+a_2-a_4, \tag{2}$$

and

$$c_{15} \equiv c_{12}+a_2-a_5. \tag{3}$$

From Cone 2 we find that b_6+a_2 is residual with a_3+c_{23}, and also with a_1+c_{12}. Hence these are coresidual, that is

$$c_{23} \equiv c_{12}+a_1-a_3. \tag{4}$$

So also

$$c_{24} \equiv c_{12}+a_1-a_4, \tag{5}$$

and

$$c_{25} \equiv c_{12}+a_1-a_5. \tag{6}$$

Similarly Cone 3 leads to $a_4+c_{43} \equiv a_1+c_{13}$.

Making use of relation (1), it follows that

$$a_4+c_{43} \equiv c_{12}+a_1+a_2-a_3,$$

and we shall agree to mean the same thing by

$$c_{43} \equiv c_{12}+a_1+a_2-a_3-a_4. \tag{7}$$

So also

$$c_{53} \equiv c_{12}+a_1+a_2-a_3-a_5. \tag{8}$$

From Cone 4 we obtain $a_5+c_{45} \equiv a_1+c_{14}$,

that is

$$c_{45} \equiv c_{12}+a_1+a_2-a_4-a_5. \tag{9}$$

Lastly, from Cone 5, we get

$$a_5+b_6 \equiv c_{12}+c_{34}; \tag{10}$$

and we have already obtained the symbol for c_{34}. We have thus expressed the remaining ten lines in terms of the six that were chosen. Now $c_{12}+a_1+a_2$ occurs frequently in the above relations. It is residual with b_6. Also b_6+u is a hyperplane section, if u is a cubic lying in any hyperplane through b_6. Hence

$$c_{12}+a_1+a_2 \equiv u.$$

Now the above ten relations may be written as follows, in terms of $a_1\,a_2\,a_3\,a_4\,a_5$ and u,

$$c_{hk} \equiv u-a_h-a_k, \qquad b_6 \equiv 2u-s,$$

where

$$s = \sum_1^5 a_r.$$

We have thus shown that the lines on Γ can be expressed in terms of five non-intersecting lines $a_1\,a_2\,a_3\,a_4\,a_5$ and a cubic u which does not meet any of these five lines. It follows, therefore, from what we said at the end of the last section, that any curve on Γ may be represented by a symbol of the form $\lambda u+\Sigma\lambda_r a_r$, where the λ's are all integers. The six elements u and a_r may be called the six fundamental elements. This is the completed statement of the theorem proved in § 9.

12. The ten fundamental conics are given by

$$c_{h6} = u-a_h, \qquad b_h = 2u-s+a_h.$$

We have seen above that every other conic on Γ must be capable of being expressed by one or other of these ten symbols. We remarked in § 10 that any two hyperplane sections are coresidual. In particular b_h+c_{h6} is a hyperplane section. Hence every hyperplane section is coresidual with $3u-s$.

Any hyperplane through a line l on Γ meets Γ in a cubic curve; and this is the only way in which we can obtain cubics on Γ. Hence any cubic on Γ is represented by a symbol of the form $3u-s-l$. Since l may take any one of the three forms a_r, $u-a_p-a_q$, $2u-s$, it follows that there are sixteen types of cubics given by $3u-s-a_r$, $2u-s+a_p+a_q$, u; there are five types of the first form, ten of the second, and one of the third. Lastly, we observe that the section of Γ by any hyperplane is a quartic curve of the first species; conversely every such quartic is a hyperplane curve, and is given by $3u-s$.

13. We now proceed to find the number of intersections of two curves given by $\lambda u+\Sigma\lambda_r a_r$, and $\mu u+\Sigma\mu_r a_r$. This number can be found if we

know the number of intersections of every two of the six fundamental elements, and adopt an appropriate convention as to what is to be meant by "the number of intersections of a line with itself". We saw in § 6 that two cubics, both of which are residuals of the same line, meet in only one point; and that any cubic is met by its residual line in two points, and is not met by any of the five secants of this line. We also saw in § 8 that the cubic u is the residual of b_6. Hence while two cubics u meet in only one point, a cubic u is not met by any of the lines a_r; nor do these lines intersect among themselves. Thus the number of intersections of the two curves is

$$N = \lambda\mu[uu]+\Sigma\lambda_r\mu_r[a_ra_r].$$

Here $[uu]$ stands for the number of points common to two curves u, and we have seen that this number is one. $[a_ra_r]$ stands for the number of intersections of a_r with itself. It is obvious on grounds of symmetry that the meaning to be attached to the symbol $[a_ra_r]$ should be such that it does not depend upon r; hence

$$N = \lambda\mu+[a_ra_r]\Sigma\lambda_r\mu_r.$$

It remains to ascertain the meaning to be attached to $[a_ra_r]$. Consider now the following particular cases:—

(1) b_6 meets a_1 in one point; and $N = -[a_1a_1]$:

(2) b_2 meets a_1 in one point; and $N = -[a_1a_1]$:

(3) c_{16} meets b_2 in one point; and $N = 2+[a_1a_1]$:

(4) the two cubics $2u-s+a_4+a_5$ and $3u-s-a_3$ meet in two points; here $N = 6+4[a_1a_1]$.

These examples suggest that $[a_1a_1]$ should be taken to stand for -1; or in other words, that the line a_r shall be considered to intersect itself in -1 points. We shall agree to make this assumption. Then we obtain

$$N = \lambda\mu-\Sigma\lambda_r\mu_r.$$

In particular, two curves of the system $\lambda u+\Sigma\lambda_r a_r$ meet in $\lambda^2-\Sigma\lambda_r^2$ points; and this number is called the grade n of curves of this system. As a second example, consider the number of intersections of the curve $\lambda u+\Sigma\lambda_r a_r$ with the curve $3u-s$. The number is $3\lambda+\Sigma\lambda_r$, and this is called the order m of curves of this system because $3u-s$ is a hyperplane section.

14. We now propose to show that the order and grade of a curve are the same as the order and grade of its projection from an arbitrary point

on to an arbitrary hyperplane. This is obvious about the order, for the projected curve is met by any plane situated in the hyperplane of projection in the same number of points as the original curve is met by an arbitrary hyperplane.

Further, the system of curves residual with a given curve c_0 on Γ is given by a linear series. On projection the linear series is transformed into a linear series; hence the number of intersections of two curves of the system remains unaltered. In other words, the grade remains.*

We now define the genus π of a curve in space of four dimensions as the genus of its projection from an arbitrary point on to an arbitrary hyperplane. All these projected curves have the same genus in the usual sense of the word, because they are all in (1, 1)-correspondence with the original curve, and therefore also amongst themselves. Prof. Baker has shown† that if m', n', π' are the order, grade, and genus of a curve on the cubic surface, then

$$n' = 2(\pi' - 1) + m'.$$

Our definition of the genus allows us to take the centre of projection anywhere we like. We can therefore take it on Γ itself without any loss of generality. We have seen that $m' = m$, and that $n' = n$; and by definition $\pi' = \pi$. Hence it follows that the genus of any curve on Γ is given by

$$\pi = 1 + \tfrac{1}{2}(n - m).$$

It may be remarked that the same formula must also hold in the case of a quartic surface with a double conic.

15. We proceed to investigate the range of variation of some of the numbers associated with the curve $\lambda u + \Sigma \lambda_r a_r$. We can obtain all the irreducible curves of given order m and given genus π by solving in integers the two equations

$$3\lambda + \Sigma\lambda_r = m, \quad \lambda^2 - \Sigma\lambda_r^2 = 2(\pi - 1) + m = n.$$

Plainly λ^2 must always be greater than or equal to n. The lower limit for λ is thus $\sqrt{(n+\epsilon)}$, where ϵ is the smallest positive (or zero) integer which makes $n+\epsilon$ a square. We may therefore write

$$\lambda = \sqrt{(n+\epsilon)} + p,$$

where p is a positive (or zero) integer. There is no loss of generality in taking λ as positive.

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* See Baker, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, 1913, p. 10.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1912, p. 290.

We can now obtain an upper limit for p; for the equations may be written in the form

$$\Sigma\lambda_r = m-3\sqrt{(n+\epsilon)}-3p, \quad \Sigma\lambda_r^2 = p^2+\epsilon+2p\sqrt{(n+\epsilon)}.$$

It is obvious that if we fix the sum of the five λ's, the sum of their squares is least when they are equal. Hence we must have

$$[m-3\sqrt{(n+\epsilon)}-3p]^2 \leqslant 5\,[p^2+\epsilon+2p\sqrt{(n+\epsilon)}],$$

which leads to $$p \leqslant \frac{3m}{4} - \sqrt{(n+\epsilon)} + \frac{\sqrt{5}}{2}\sqrt{(m^2-4n)}.$$

There are thus only a finite number of solutions in each case. The limit obtained for p is only the maximum limit, and p may not range right up to the limit; this is because of the fractions involved in the division by 5. But in practice we start with the value $\sqrt{(n+\epsilon)}$ for λ, and proceed step by step till we reach the value λ' for which $(m-3\lambda')^2 > 5\,(\lambda'^2-n)$.

The use we make of the limiting value obtained for p is the following. The expression involves the surd $\sqrt{(m^2-4n)}$. It is obvious, therefore, that $m^2 \geqslant 4n$. This enables us to set an upper limit for the genus of curves of given order on Γ. In fact

$$\pi = 1+\tfrac{1}{2}\,(n-m),$$

and hence $$\pi \leqslant \left(1-\frac{m}{4}\right)^2+\left(\frac{m}{4}\right)^2.$$

For example, for quintics on Γ we must have $\pi \leqslant 1\frac{5}{8}$, showing that there are only two types of quintics on Γ, those of genus zero and those of genus one.

16. We now proceed to apply this method to obtain all the curves on Γ up to the fifth order.

For lines we have $m = 1$, $\pi = 0$, $n = -1$, $\epsilon = 1$, and $p \leqslant 3$. The equations to solve are

$$3\lambda+\Sigma\lambda_r = 1, \quad \lambda^2-\Sigma\lambda_r^2 = -1.$$

The solutions are typified by

$$\lambda = 0, \quad \lambda_1 = 1, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0;$$

$$\lambda = 1, \quad \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = \lambda_4 = \lambda_5 = 0;$$

$$\lambda = 2, \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -1.$$

These lead to the sixteen lines a_r, $u-a_r-a_s$, $2u-s$.

For conics we take $m = 2$, $\pi = 0$, $n = 0$, $\epsilon = 0$, and $p \leqslant 3$. The

equations are $$3\lambda+\Sigma\lambda_r=2, \quad \lambda^2-\Sigma\lambda_r^2=0.$$

The solutions are typified by

$$\lambda=1, \quad \lambda_1=-1, \quad \lambda_2=\lambda_3=\lambda_4=\lambda_5=0;$$

and $$\lambda=2, \quad \lambda_1=0, \quad \lambda_2=\lambda_3=\lambda_4=\lambda_5=-1.$$

These lead to the ten types of conics $u-a_r$, $2u-s+a_r$. Any conic $u-a_r$ is thus in a hyperplane with any conic $2u-s+a_r$.

For cubics we have $m=3$, $\pi=0$, $n=1$, $\epsilon=0$, and $p\leqslant 3$. The equations to solve are

$$3\lambda+\Sigma\lambda_r=3, \quad \lambda^2-\Sigma\lambda_r^2=1.$$

The solutions are typified by

$$\lambda=1, \quad \lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=0;$$

$$\lambda=2, \quad \lambda_1=\lambda_2=0, \quad \lambda_3=\lambda_4=\lambda_5=-1;$$

$$\lambda=3, \quad \lambda_p=-2, \quad \lambda_r=0 \ (r\neq p).$$

These lead to the sixteen types of cubics given by

$$u, \tag{A}$$

$$2u-a_1-a_2-a_3, \tag{B}$$

$$3u-s-a_3. \tag{C}$$

The relations established in § 6 can all be verified by means of these expressions. For instance, every cubic has a bisecant, the three curves put down having b_6, c_{45}, and a_3 as bisecants.

For quartics of the first species we take $m=4$, $\pi=1$, $n=4$, $\epsilon=0$, and $p\leqslant 5$. The equations are

$$3\lambda+\Sigma\lambda_r=4=\lambda^2-\Sigma\lambda_r^2.$$

The only solution that exists leads to the curves $3u-s$, showing that all such quartics are hyperplanar curves. It follows that a quartic of the first species is met by any curve of order p on Γ in p points.

For quartics of the second species we take $m=4$, $\pi=0$, $n=2$, $\epsilon=2$, and $p\leqslant 4$. The equations are

$$3\lambda+\Sigma\lambda_r=4, \quad \lambda^2-\Sigma\lambda_r^2=2.$$

The solutions are typified by

$$\lambda=2, \quad \lambda_1=\lambda_2=-1, \quad \lambda_3=\lambda_4=\lambda_5=0;$$

$$\lambda=3, \quad \lambda_1=0, \quad \lambda_2=-2, \quad \lambda_3=\lambda_4=\lambda_5=-1;$$

$$\lambda=4, \quad \lambda_1=\lambda_2=-1, \quad \lambda_3=\lambda_4=\lambda_5=-2.$$

There are thus forty types of quartics typified by

$$2u-a_1-a_2, \tag{A}$$

$$3u-2a_1-a_2-a_3-a_4, \tag{B}$$

$$4u-a_1-a_2-2a_3-2a_4-2a_5. \tag{C}$$

The curves A and C lie on a quadratic; so do two curves B properly chosen. Each of the curves is the coresidual of a cubic and a line; for instance, the three curves written down are coresidual with a cubic $A+c_{12}$, a cubic $B+c_{14}$, and a cubic $C+c_{45}$, respectively. Hence every rational quartic is coresidual with a cubic and a line which does not meet its bisecant.

For quintics of genus zero we have $m=5$, $\pi=0$, $n=3$, $\epsilon=1$, and $p\leqslant 9$. The curves are typified by

$$2u-a_5, \tag{A}$$

$$3u-a_1-a_2-2a_3, \tag{B}$$

$$4u-s-2a_1, \tag{C}$$

$$4u-a_2-2a_3-2a_4-2a_5, \tag{D}$$

$$5u-2s-a_1+a_5. \tag{E}$$

There are thus eighty classes. It can be verified that each of these curves is the residual of a group of three lines of which only two intersect: and there are 240 such groups of which only 80 are effectively distinct.*

For quintics of genus one we take $m=5$, $\pi=1$, $n=5$, $\epsilon=4$, $p\leqslant 3$. The curves are of sixteen types given by

$$3u-s+a_3, \tag{C}$$

$$4u-s-a_4-a_5, \tag{B}$$

$$5u-2s. \tag{A}$$

These quintics A, B, C lie on quadratics with the cubics A, B, C respectively.

17. We are now in a position to answer the question raised at the end

* For instance, the three groups $(c_{23},\ a_1,\ b_6)$, $(c_{23},\ c_{25},\ c_{34})$, and $(c_{23},\ c_{24},\ c_{35})$ are symbolically equal to one group only.

of § 6. But we shall first give a discussion of the concept of a double six on Γ.

An arbitrary point O on Γ is fixed first. This determines uniquely ten conics through O, and a zero circle at O. These taken together with the sixteen lines on Γ make up twenty-seven elements. A double six on Γ is now defined as consisting of two rows of six each, chosen from the twenty-seven in such manner that their mutual relations as to intersections are similar to those in the case of an ordinary double six. We introduce the single condition that the point O itself shall not be counted among the intersections. Then the two concepts are exactly identical, except for the fact that the double six on Γ includes among its elements conics as well as lines. Any element in either row meets all the elements of the other row, excepting the one opposed to it; and the elements of a row do not intersect among themselves.

A single six being understood to mean one half of a double six, consider the composition of the single sixes on Γ in terms of the zero circle, the ten conics, and the sixteen lines.

(A) There are sixteen, each involving one line and five conics. Types are

$$a_1\ b_1\ c_{26}\ c_{36}\ c_{46}\ c_{56}, \tag{1}$$

$$b_6\ b_1\ b_2\ b_3\ b_4\ b_5, \tag{2}$$

$$c_{12}\ c_{16}\ c_{26}\ b_3\ b_4\ b_5. \tag{3}$$

There are five of the first type, one of the second, and ten of the third. In each case the first element is the line and the other five are conics.

(B) There are sixteen, each involving the zero circle and five lines. Types are

$$a_6\ b_6\ c_{12}\ c_{13}\ c_{14}\ c_{15}, \tag{1}$$

$$a_6\ a_1\ a_2\ a_3\ a_4\ a_5, \tag{2}$$

$$a_6\ a_2\ a_1\ c_{45}\ c_{53}\ c_{31}. \tag{3}$$

There are five of the first type, one of the second, and ten of the third. In each case the first element is the zero circle, the rest being lines. The single sixes A_1, B_1 form a double six; so also do A_2, B_2 and A_3, B_3.

(C) There are forty, each involving two conics and four lines, ten of

each of the following four types :—

$$a_4\ c_{15}\ c_{25}\ c_{35}\ b_4\ c_{65}, \tag{1}$$

$$a_5\ c_{14}\ c_{24}\ c_{34}\ b_5\ c_{64}, \tag{2}$$

$$a_1\ a_2\ a_5\ c_{34}\ c_{46}\ c_{63}, \tag{3}$$

$$c_{25}\ c_{15}\ c_{12}\ b_6\ b_3\ b_4. \tag{4}$$

C_1, C_2 form a double six ; so also do C_3, C_4. We have thus obtained thirty-six double sixes on Γ, and there are no more.

18. Any cubic in a hyperplane through l, one of the sixteen lines on Γ, meets l in two points. There is thus a family of cubics associated with each of the single sixes A. It can be shown by elementary considerations that any cubic meets the other five elements of the associated single six in two points each, does not meet the elements of the complementary single six from B, and meets each of the fifteen other elements in one point each.

In particular, consider the single sixes A_3, B_3. The family of cubics thus associated with A_3 is given by $3u-s-c_{12}$. The curves coresidual with $b_6+c_{16}+c_{26}$ are similarly associated with B_3; but these are quintics with a double point at O, and are given by $4u-s-a_1-a_2$. Similarly A_2 leads to the cubics u ; and B_2 to curves coresidual with b_6 and the section by any hyperplane passing through the plane of the zero circle. These curves are quintics passing twice through O, and are given by $5u-2s$. So also A_1, B_1 lead to the cubics $3u-s-a_1$, and the quintics passing twice through O given by $3u-s+a_1$. The curves associated with the single sixes A, B are thus the sixteen systems of cubics and the sixteen systems of quintics with a double point at O. From the expressions found above, it is seen that any cubic associated with a single six lies on a quadratic with any quintic associated with the complementary single six. The curves similarly associated with the single sixes C are quartics through O given by

$$3u-a_1-a_2-a_3-2a_4,$$

$$3u-a_1-a_2-a_3-2a_5,$$

$$4u-2a_1-2a_2-a_3-a_4-2a_5,$$

$$2u-a_3-a_4.$$

Any two curves of the first two types lie on a quadratic; so also do any two curves of the third and fourth types.

19. We have thus obtained sixteen systems of cubics, sixteen of quintics, and forty of quartics. The quartics pass through O, and the quintics have a double point at O. The seventy-two systems of curves, therefore, project from O into the seventy-two systems of cubic curves on the cubic surface S_3.

Two corresponding quartics lie on a quadratic; so also do a cubic and the corresponding quintic. We thus obtain thirty-six systems of quadratics associated with the thirty-six double sixes. We will now show that these quadratics are all hypercones with O for vertex. The curve of intersection of Γ and the quadratic is a composite curve consisting of either two quartics each passing through O, or a cubic and a quintic with a double point at O. In either case, therefore, the composite curve has a double point at O. This can only happen if the quadratics are hypercones.

Now a hypercone with vertex at O will contain a cubic entirely if it passes through seven of its points, for a cubic is a hyperplanar curve. Further, a hypercone is determined, given the vertex, by nine points more. Hence the equation to the general hypercone vertex O passing through a particular cubic is of the form

$$C_0+\lambda_1 C_1+\lambda_2 C_2 = 0,$$

where C_0, C_1, C_2 are three particular such hypercones. The general hypercone, with vertex at O, through a quintic passing twice through O, must also be of the same form. All such hypercones can therefore be represented as linear functions of nine of them. Hence all the hypercones associated with any one double six A, B are harmonically circumscribed to a unique hypercone with vertex at O. This hypercone corresponds to Schur's quadric and may be called Schur's cone. A similar argument holds about the quartics and the double sixes C. Thus there are thirty-six Schur's cones.

20. We shall now prove that if $A_pA_qA_rA_sA_t$ be the five points of intersection of a cubic and a quintic, then the polar line of the plane $A_pA_qA_r$ with reference to the associated Schur's cone lies on the plane OA_sA_t. For consider the hyperplane through the five points; they are all in a hyperplane because the cubic lies in a hyperplane. The quintic projects from O into a cubic on this hyperplane. We have thus two cubics in this hyperplane meeting in five points and lying on a quadric C', which is harmonically circumscribed to the quadric C common to Schur's hypercone and the hyperplane. It is known* that the pole of the plane

* Reye, *loc. cit.*, p. 258.

$A_pA_qA_r$ with reference to C lies on the line A_sA_t. Hence the statement is proved; a somewhat similar argument holds in the case of the quartics. It follows from this, precisely as in the three-dimensional case, that with reference to a Schur's cone the elements of the corresponding double six are connected as follows:—

(1) If the opposed elements are both lines, then each has for polar plane the plane joining O to the other.

(2) If both are conics, the plane of each contains the polar line of the plane of the other.

(3) If the elements are a line and a conic, the plane of the conic is the polar plane of the line; and the polar line of the plane joining O to the line lies in the plane of the conic. This includes the case where the elements are a line and the zero circle.

21. A reference to Prof. Baker's paper referred to in the introduction will make it plain that the entire theory of coresiduation on Γ could have been deduced as a generalisation of his work. In fact, given the symbol of any curve on the one surface, we can obtain the symbol of the corresponding curve on the other surface. To the zero circle at O on Γ corresponds the line a_6 on S_3. Hence, if a curve on Γ passes k times through O, the projected curve meets a_6 k times. In other words, if $\lambda u+\Sigma\lambda_r a_r$ is any curve on Γ, and if further we know that it has a multiple point of order $-\lambda_6$ at O, then the projected curve is given by

$$\lambda u+\sum_1^6 \lambda_r a_r.$$

This is capable of easy verification.

22. We now proceed to discuss the representation of Γ on a plane R_2. Any plane meets a quadratic in a conic: in particular, any plane through b_6 meets F and Φ in one other line on each. These two lines meet in a point on Γ. Also an arbitrary plane meets R_2 in a point. Thus we can establish a (1, 1)-correspondence between Γ and R_2 by means of planes through b_6.

To the five lines a meeting b_6 there correspond five points A_r on R_2. The hyperplane through b_6 and c_{hk} meets R_2 in a line c'_h which is the projection of c_{hk}. Since c_{hk} meets both a_h and a_k, it follows that c'_{hk} passes through A_h and A_k. Thus to the ten lines c_{hk} correspond the ten lines

$A_h A_k$. Further, u is a cubic lying in a hyperplane through b_6. Any such cubic therefore projects into a line, viz. the line of intersection of R_2 with the hyperplane. Conversely, any arbitrary line in R_2 is the projection of a cubic of the system u.

Since b_6 meets the five lines a_r, it follows that the projection of b_6 is a locus passing through the five points A_r. We have seen that any cubic u meets b_6 in two points. Hence the image of b_6 is a locus met by any line in two points. Thus b_6 projects into the conic passing through the five points A_r. The plane representation of Γ has therefore the five points A_r and an arbitrary line u as the fundamental elements.

We now proceed to interpret some of the symbols we have obtained in coresiduation. The number of points in which any curve $\lambda u + \Sigma \lambda_r a_r$ is met by the cubic u is the same as the number of points in which the projected curve is met by an arbitrary line; in other words, λ is the order of the projected curve. Further, $\lambda u + \Sigma \lambda_r a_r$ meets the line a_r in $-\lambda_r$ points. Hence the projected curve has a multiplicity of order $-\lambda_r$ at A_r. From this it follows that if we suppose λ positive, as we may without loss of generality, then the five numbers λ_r are all negative.

The projected curve is of order λ; and, in general, its only multiple points are at the points A_r, where its multiplicity is of order $-\lambda_r$. Hence the genus of the projected curve is given by

$$\begin{aligned} 2\pi &= (\lambda-1)(\lambda-2) - \Sigma(-\lambda_r)(-\lambda_r-1) \\ &= (\lambda^2 - \Sigma\lambda_r^2) - (3\lambda + \Sigma\lambda_r) + 2 \\ &= n - m - 2. \end{aligned}$$

This affords a verification that the genus remains unaltered.

We have all along made the tacit assumption that any curve on Γ can be represented by only one symbol of the form $\lambda u + \Sigma \lambda_r a_r$. The truth of this assumption is now obvious; for it follows, from what we said above about the relations of the numbers λ, λ_r to the plane curve, that these numbers are determined uniquely as soon as the curve is prescribed.

We are now in a position to interpret all the symbolism of coresiduation on Γ with reference to the representation of Γ on the plane. Any hyperplane section projects into a cubic curve through the points A_r. The system of conics $u - a_r$ becomes the system of lines through A_r; and the system $2u - s + a_r$ projects into conics through the four points A_k, where k is not equal to r. Through any point O on Γ there pass ten conics lying on the surface. But, if the point lies on one of the lines such as b_6, there are only five non degenerate conics. Thus, if we fix a point f

on b_6, the five conics of the type $u-a_r$ that pass through f project into the five lines FA_r, where F is the point on the conic corresponding to f. We could now generalise some of the properties associated with Pascal's configuration, but we proceed to consider coresiduation on the quartic with a double conic.

PART III.

23. We propose to project Γ from an arbitrary point O to an arbitrary hyperplane R_3. There is one member of the network $F+\lambda\Phi=0$ which passes through O. The tangent hyperplane at O to this quadratic meets it in a quadric cone with O for vertex. This quadric cone meets Γ in a quartic curve which projects into the conic in which R_3 meets the cone. This is the double conic. The analytical investigation is interesting and may be exhibited as follows.

Taking O as $(0, 0, 0, 0, 1)$, and R_3 as $u=0$, the equations defining Γ may be taken in the form

$$F=u^2+2uf_1+f_2=0, \qquad \Phi=u^2+2u\phi_1+\phi_2=0,$$

where f_1, ϕ_1 are homogeneous linear functions and f_2, ϕ_2 are homogeneous quadratic functions of x, y, z, t. The coordinates on R_3 are given by

$$X=x, \quad Y=y, \quad Z=z, \quad T=t.$$

Hence the projected surface has for equation the eliminant of u between F and Φ with x, y, z, t replaced by X, Y, Z, T. The surface is thus

$$(F_2-\Phi_2)^2=4(F_1-\Phi_1)(F_2\Phi_1-F_1\Phi_2), \tag{1}$$

where the capital letters denote the same functions of X, Y, Z, T as the corresponding small letters are of x, y, z, t. The quadratic through O is

$$2u(f_1-\phi_1)+(f_2-\phi_2)=0.$$

Its tangent hyperplane at O is $f_1-\phi_1=0$; and the quadric cone with O for vertex is $f_2-\phi_2=0$. The double conic is thus the projection of the quartic curve of intersection of this quadric cone with Γ.

The equation (1) may be written in the form

$$\Theta^2=4W^2\Psi,$$

where $$\Theta = (F_2 - \Phi_2) - 2F_1(F_1 - \Phi_1),$$

$$W = (F_1 - \Phi_1),$$

and $$\Psi = (F_1^2 - F_2),$$

showing that it is a quartic with a double conic. It may be remarked that if we replace f_1, ϕ_1, f_2, ϕ_2 by f_1/λ, f_2/λ, ϕ_1/λ, ϕ_2/λ respectively, and then put $\lambda = 0$, we obtain the projection of Γ from a point on itself; and this surface is the cubic surface

$$F_2\Phi_1 = F_1\Phi_2.$$

Conversely, given the equation of the surface in the form

$$\Theta^2 = 4W^2\Psi,$$

we can regain the two equations determining Γ by means of the transformation

$$\frac{WX}{x} = \frac{WY}{y} = \frac{WZ}{z} = \frac{WT}{t} = \frac{\mathrm{X}}{u},$$

where X is a properly chosen quadratic function of X, Y, Z, T. The relation $\Theta^2 = 4W^2\Psi$ leads to a similar relation $\theta^2 = 4w^2\psi$ between x, y, z, t.* Now suppose we take

$$\mathrm{X} = -\tfrac{1}{2}(\Theta + 2F_1W).$$

Eliminating Θ between this equation and

$$\Theta^2 = 4W^2\Psi,$$

we obtain $$(\mathrm{X} + F_1W)^2 = W^2\Psi,$$

which is the same as $$\mathrm{X}^2 + 2\mathrm{X}F_1W + F_2W^2 = 0,$$

since $$F_2 = F_1^2 - \Psi.$$

From this we deduce the relation

$$u^2 + 2uf_1 + f_2 = 0,$$

which together with $\theta^2 = 4w^2\psi$ defines Γ.

24. We proceed to show that the two following definitions of residuation on S_4† are identical in effect.

* It should be added that, in this article and the next, capital letters and small letters indicate precisely the same functions of the variables X, Y, Z, T and x, y, z, t respectively.

† S_4 is the quartic with a double conic. We shall use the word cyclide in the general sense of a quartic surface with a double conic; and we shall denote the double conic by D_2.

(A) Two curves on the cyclide are said to be coresidual if the corresponding curves on Γ are coresidual.

(B) Two curves A and B on the cyclide are said to be coresidual if there exists a curve C such that one surface can be drawn to pass through A, C, and D_2, and another of the same order to meet the cyclide in B, C, D_2.

The complete identity of the two definitions will follow from the following theorem.

If through two curves on the cyclide and D_2 we can draw a surface of order $r+1$, then we can draw a variety of order r through the corresponding two curves on Γ. Conversely, if through two curves on Γ we can draw a variety of order r, then through the two corresponding curves on the cyclide and D_2 we can draw a surface of order $r+1$. The proof is as follows.

Taking D_2 as $\Theta = W = 0$, where Θ is a quadric and W a plane, any surface of order $r+1$ passing through D_2 has an equation of the form

$$\Theta A_{r-1} = WB_r;$$

A and B are homogeneous functions of X, Y, Z, T, the suffix indicating the order. There is then, obviously, a conical variety of order $r+1$ given by $\theta a_{r-1} = wb_r$ of which O is the vertex. This is a variety meeting Γ in the two corresponding curves and the quartic which projects into D_2. Since $\Theta = W = 0$ is the double conic, it follows that $\theta = w = 0$ is the hyperplane section that projects into D_2. Also $w = 0$ is the same as $f_1 - \phi_1 = 0$.

Thus $(F - \Phi - \theta)\, a_{r-1} = wb_r$ is a variety of order $r+1$ passing through the curve of intersection of

$$\theta a_{r-1} = wb_r \qquad F = \Phi = 0.$$

But
$$\theta = (f_2 - \phi_2) - 2f_1 (f_1 - \phi_1).$$

Hence writing out F and Φ in full, and making use of this last relation, we obtain

$$2(u + f_1)(f_1 - \phi_1)\, a_{r-1} = wb_r.$$

But $w = f_1 - \phi_1$. Hence this equation takes the form

$$\{2(u + f_1)\, a_{r-1} - b_r\}\, w = 0,$$

which represents the hyperplane $w = 0$ and a variety of order r. Thus the first part of the theorem is proved.

Conversely, let V_r be a variety of order r through the two given curves on Γ, given by

$$V_r = u^r + u^{r-1}a_1 + \ldots + a_r = 0.$$

Replacing every u^2 that occurs here by $-(2u\phi_1 + \phi_2)$, we ultimately obtain an expression of the form $ua_{r-1} + a_r$. If now we replace u by $-\frac{f_2 - \phi_2}{2w}$, that is, by $-\frac{\theta + 2f_1 w}{2w}$, we obtain

$$(\theta + 2f_1 w)\, a_{r-1} = 2a_r w.$$

This is obviously a conical variety of order $r+1$, with O for vertex, and passing through the intersection of V_r with Γ, besides the hyperplane section $\theta = w = 0$ which yields D_2. The intersection of this conical variety with the hyperplane R_3 gives us a surface of order $r+1$ passing through D_2 and the corresponding two curves on the cyclide. This completes the proof of the theorem. Hence the two definitions of coresiduation on the cyclide that we have given are entirely identical.

25. We now proceed to deduce the formulæ for coresiduation on S_4 from the formulæ established in Part II. Since Γ and S_4 are in (1, 1)-correspondence, the same set of fundamental elements holds for S_4. All the symbolism holds valid. But the significance of the symbol $3u - s$ calls for discussion. With reference to the origin of projection O, all hyperplane sections of Γ may be divided into two classes, those passing through O and those not passing through O. The projections are thus either plane sections of S_4 or twisted quartics. Both these are represented by the symbol $3u - s$. This is a phenomenon which arises whenever the locus we are considering may be obtained as the projection of a locus of the same order in space of higher dimensions.

For instance, in the present case, the general hyperplane $\Sigma ax = 0$ projects into

$$W(aX + bY + cZ + dT) + e(-\tfrac{1}{2}\Theta - F_1^1 W) = 0.$$

This is in general a quadric surface, and reduces to a plane $\Sigma aX = 0$ only when $e = 0$. It may be remarked that the quadric represented by this equation when e is not zero passes through the double conic $\Theta = W = 0$. Thus any plane section of S_4 is coresidual with the residual curve of intersection of a quadric through D_2, and this is a quartic of the first

species. A plane section of S_4 is thus an example of an incomplete series, the complete series being given by quadrics through D_2.

Again, consider two curves C_m and $C_{m'}$ on S_4, of orders m and m', given by

$$\lambda u + \Sigma \lambda_r a_r$$

and

$$(3p-\lambda)\, u + \Sigma(-\lambda_r - p)\, a_r,$$

$m+m'$ being equal to $4p$. From our definitions it follows that the only logical conclusion is that C_m and $C_{m'}$ lie on a surface of order $p+1$ through D_2. In particular cases this surface may degenerate into the plane of D_2 and a surface of order p. In some of the simpler cases the following considerations will enable us to decide if such a degeneration is possible. Suppose, in fact, that a surface of order p is possible. Then this surface is met by the plane of D_2 in a curve of order p which meets D_2 in $2p$ points. But C_m and $C_{m'}$ meet the plane of D_2, and hence D_2 itself in m and m' points respectively; but $m+m' = 4p$. This can only happen if these $4p$ points consist of the $2p$ points each taken twice.

26. We now proceed to an analysis of the curves on the cyclide. Any curve on S_4 may be represented by a symbol of the form

$$\lambda u + \sum_1^5 \lambda_r a_r,$$

where u is a cubic and the lines a_r are five non-intersecting lines. The other eleven lines are given by

$$c_{hk} = u - a_h - a_k, \quad b_6 = 2u - s,$$

where

$$s = \Sigma a_r.$$

The conics on S_4 are of ten types, and are given by $2u-s+a_p$ and $u-a_p$. Any plane section is $3u-s$; but, as we have seen, this symbol also represents other curves.

We will now show that the formulæ for the order, grade, and genus remain unaltered. It is obvious that the order remains the same. The grade remains unchanged because projection is only a birational transformation. We have seen that on Γ the genus, grade, and order are connected by the relation

$$2(\pi-1)+m = n.$$

Since the order and grade remain unaltered, it follows from our definition of the genus of a curve in space of four dimensions that the order, grade and genus of a curve on S_4 are connected by the same relation.

The double conic D_2 itself must be represented by $\frac{3}{2}u-\frac{1}{2}s$. The number of intersections of two curves $\lambda u+\Sigma\lambda_r a_r$, and $\mu u+\Sigma\mu_r a_r$ is $\lambda\mu-\Sigma\lambda_r\mu_r$. When one of these curves is D_2 itself, the result given by this formula is always one-half of an integer; we shall agree to take the number of intersections as double the result given by the formula. With this interpretation, we see that any curve of order p on S_4 meets D_2 in p points.

The conics on S_4 are of ten types; two conics of the type $u-a_p$ do not meet, nor do two conics of the type $2u-s+a_p$. But a conic of the type $u-a_p$ meets a conic of the type $2u-s+a_p$ in two points unless they lie in the same plane, in which case they have four points in common.

Plane cubics are given by $3u-s-l$, where l is any line on the surface; and any cubic $3u-s-l$ meets l in two points. We see from § 16 that there are sixteen classes of twisted cubics typified by

$$u, \tag{A}$$

$$2u-a_1-a_2-a_3, \tag{B}$$

$$3u-s-a_1. \tag{C}$$

There are thus $16\infty^2$ of twisted cubics on S_4, for there are ∞^2 hyperplanes through any line. Two cubics of the same class meet in only one point; but types (A) and (C) meet in three points, and types (A) and (B) in two points. Each of these classes is associated with one line on the surface which is a bisecant* of all curves of the class. The three cubics written down have b_6, c_{45}, and a_1 for bisecants. In each case, the ten lines not meeting the bisecant are unisecants and the five lines meeting the bisecant are zerosecants. Lastly, the bisecant is in each case the residual intersection with S_4 of a quadric through the cubic and D_2.

Twisted quartics of genus one are given by $3u-s$. But this is the symbol for a plane section. Hence, leaving aside the case of a plane quartic, which is trivial, we reach the result that the residual intersection with S_4 of any quadric through D_2 is a quartic curve of genus one which meets any curve on S_4 of order p in p points. Two such quartics meet in four points lying in a plane, for two quadrics through D_2 have their remaining points of intersection on a plane. Through any four nonplanar points of S_4 we can draw one such quartic, for through the four points

* By zerosecant, unisecant, bisecant, &c., we mean lines on the surface S_4 which meet the curve in question in 0, 1, 2, &c. points

and D_2 we can draw one quadric. There are ∞^4 twisted quartics of the first species, for there are ∞^4 hyperplanes in space of four dimensions.

27. From § 16 it follows that there are forty classes of rational quartics on S_4, ten of each of the following four types

$$3u - a_1 - a_2 - a_3 - 2a_4, \qquad (A_1)$$

$$3u - a_1 - a_2 - a_3 - 2a_5, \qquad (A_2)$$

$$4u - 2a_1 - 2a_2 - 2a_3 - a_4 - a_5, \qquad (B_1)$$

$$2u - a_4 - a_5. \qquad (B_2)$$

The residual intersection with S_4 of the quadric through any rational quartic must also be a rational quartic.* Hence it follows that any curve of the type (A_1) lies on a quadric together with a properly chosen curve of the type (A_2). So also any curve of type (B_1) lies on a quadric together with a particular curve of type (B_2).

Each of these quartics has four bisecants. The four quartics written down have as bisecants

$$a_4 \; c_{15} \; c_{25} \; c_{35}, \qquad (\alpha_1)$$

$$a_5 \; c_{14} \; c_{24} \; c_{34}, \qquad (\alpha_2)$$

$$a_1 \; a_2 \; a_3 \; c_{45}, \qquad (\beta_1)$$

$$c_{23} \; c_{31} \; c_{12} \; b_6. \qquad (\beta_2)$$

It is necessary to remark† that from the sixteen lines on S_4 we can form twenty double fours, ten of the form (α_1), (α_2) and ten of the form (β_1), (β_2). These twenty double fours can be grouped into ten pairs such that each pair exhausts the sixteen lines. One such pair is (α_1, α_2), (β_1, β_2). The four bisecants of a rational quartic form one-half of a double four; its zero-secants form the other half of the same double four. The remaining eight lines are its unisecants. The bisecants of a quartic are the zero-secants of its complementary quartic, that is, that quartic which lies on a quadric together with the former one; and *vice versa*. Two quartics

* This and similar results to be used later can all be deduced from Ex. iv on p. 166 of Prof. Baker's *Abelian functions*.

† Some of the following results are given by Berzolari in § 4 of his article referred to in the introduction. But our point of view here is different, since we are deducing these results from Γ, and ultimately from the cubic surface and the double six thereon.

which lie on a quadric have the same unisecants, and these form a double four.

Further, there are eighty pairs of non-intersecting lines on S_4. From these we can form forty twisted quadrilaterals, each consisting of two pairs of non-intersecting lines. These quadrilaterals can again be paired such that each pair contains the eight elements that make up a double four. Any such quadrilateral has its elements adding up to $3u-s$. Hence in the system of quadrics through D_2 there are forty which meet S_4 in the forty quadrilaterals. If $\Theta^2 = W^2\Psi$ be S_4, any quadric through D_2 is $\Theta = WP$, where P is an arbitrary plane. The residual curve of intersection lies on the quadric $\Psi = P^2$. There is no loss of generality in taking Ψ as one of the Kummer cones. It follows that the forty quadrics of the type $\Psi = P^2$ all have a Kummer cone as a common enveloping cone. Therefore through the forty quadrilaterals we can draw five sets of forty quadrics such that each set has a Kummer cone for enveloping cone.

Again, the eight elements of a double four add up to $2(3u-s)$. Hence they lie on a cubic surface through D_2. If $\Theta^2 = W^2\Psi$ be S_4, there are twenty cubic surfaces of the form $\Theta P = W\Theta'$ which meet S_4 in a double four, Θ' being an arbitrary quadric and P an arbitrary plane. The residual intersection of any such cubic and S_4 lies on $\Theta'^2 = P^2\Psi$, which is also a cyclide. Thus there are twenty cyclides of the form $\Theta'^2 = P^2\Psi$ passing through the twenty double fours. All these have as a Kummer cone one of the Kummer cones of the original cyclide, for Ψ may be taken to be a cone. Therefore, through the twenty double fours on S_4 we can draw five sets of twenty cyclides of which each set has a Kummer cone common with the original cyclide.

Lastly, if we choose three lines from one half of a double four, the residual intersection of a cubic surface through these three lines, their transversal, and D_2 is a quartic which has the three lines for bisecants. It is easy to see that through D_2 and four such lines we can draw ∞^3 cubic surfaces. Thus there are ∞^3 rational quartics on S_4, and associated with each double four there are ∞^3 quadrics. These do not seem to be subject to any special relation.

28. We noticed in § 16 that there are eighty classes of rational quintics on Γ divided into five types. It is the same with S_4, and the same symbols are valid. Each of the sixteen lines is associated with five classes of curves for which it is a trisecant: b_6 is trisecant for the five classes typified by A; a_1 is trisecant for C and four classes of E; c_{12} is trisecant for three classes of B and two of D.

On the whole there are eighty classes; from the sixteen lines we can get eighty groups of three lines such that each group of three has only one point of intersection. The eighty classes of curves are associated with the eighty groups, in the sense that through a curve of any given class D_2 and the associated group we can draw a cubic surface. There are ∞^4 such cubic surfaces, and hence ∞^4 rational quintics on S_4. Each of these quintics has one trisecant, six bisecants, five unisecants, and four zero secants. The classes D and E are not mentioned in Prof. Jessop's treatise.*

Quintics of genus one are given by the three types

$$3u-s+a_1, \tag{A}$$

$$4u-s-a_1-a_2, \tag{B}$$

$$5u-2s. \tag{C}$$

Each of these curves has a zero secant; the three curves written down have a_1, c_{12} and b_6 as zero secants. In each case the five lines meeting the zero secant are bisecants, and the other ten lines are unisecants. Further, a quadric through the twisted cubic $3u-s-l$, where l is any line, meets S_4 in a quintic $3u-s+l$. Hence the quintics (A), (B), (C) lie on quadrics through the cubics A, B, C. A cubic surface through D_2, a pair of non-intersecting lines, and one of their transversals, meets S_4 in a further quintic which is of genus one. There are ∞^5 such surfaces, and hence ∞^5 such quintics.

I am unable to trace the system D which Prof. Jessop mentions.† In fact, the remark he makes holds of the systems A, B, C. Pairs of non-intersecting lines may be typified by $c_{12}+c_{13}$, a_1+a_2, b_6+c_{45}, a_1+c_{23}. These lead to two curves of the types (A), (B), (C) themselves.

A quintic $3u-s+l$ meets a cubic $3u-s-l$ in five points, and they lie on a quadric; these two curves are associated with the line l, in the sense that it is a zero secant for the quintic and a bisecant for the cubic. Through a cubic we can draw ∞^2 quadrics. Hence associated with each line on S_4 there are ∞^2 such quadrics.

29. Passing on to sextics on S_4, sextics of genus two are obtained by solving

$$3\lambda+\Sigma\lambda_r = 6, \quad \lambda^2-\Sigma\lambda_r^2 = 8.$$

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* See p. 44.

† *Loc. cit.*, p. 44. I have communicated these results to Prof. Jessop, who authorizes me to say that he now agrees with the view expressed here.

The curves are of ten classes given by

$$4u-s-a_r, \tag{A_r}$$

$$5u-2s+a_r. \tag{B_r}$$

There are five classes of each type. The unisecants of (A_1) are

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ c_{12} & c_{13} & c_{14} & c_{15} \end{vmatrix}.$$

These form a "double four" in which each element meets only the opposite one; and these are bisecants of (B_1). The remaining eight lines form another "double four"

$$\begin{vmatrix} b_6 & c_{23} & c_{24} & c_{25} \\ a_1 & c_{45} & c_{53} & c_{34} \end{vmatrix}.$$

These are bisecants of (A_1) and unisecants of (B_1). The curves (A_r), (B_r) are thus associated with the cone r, in the sense that the eight lines that arise from either class of conics of this cone are bisecants for the one curve and unisecants for the other. Two curves (A_r), (B_r) meet in ten points, and it can be shown that a cubic surface through (A_r) meets S_4 again in a sextic (B_r). Similarly a quadric through a conic $u-a_r$ meets S_4 in a curve (B_r); and a quadric through a conic $2u-s+a_r$ meets S_4 in a sextic (A_r). Hence we have the following result. If we draw two quadrics of which one passes through a conic $u-a_r$, and the other through a conic $2u-s+a_r$, then their residual intersections with S_4 are two sextics of genus two lying on a quartic through D_2.

Further, a cubic surface through D_2 and any pair of intersecting lines meets S_4 in a sextic of the type we are considering. Hence there are ∞^7 sextics of genus two on S_4.

Sextics of genus one are of four types

$$3u-a_1-a_2-a_3, \tag{A}$$

$$4u-2a_1-2a_2-a_3-a_4, \tag{B}$$

$$5u-3a_5-2a_4-2a_3-a_2-a_1, \tag{C}$$

$$6u-2s-a_4-a_5. \tag{D}$$

There are eighty classes of curves corresponding to the eighty pairs of non-intersecting lines. A quadric through any such pair of lines meets S_4 in a sextic of genus one. Also two curves of types (A), (D) or of types

(B), (C) lie on a cubic surface. Each of these curves has two trisecants, six bisecants, six unisecants, and two zero secants. Also each of the sixteen lines meets (A) and (D) together in three points; and each of them meets (B) and (C) together in three points.

Any cubic surface through D_2 and a pair of non-intersecting lines meets S_4 in a sextic of genus one. Thus there are ∞^6 sextics of genus one. Given a pair of non-intersecting lines on S_4, there is only one other such pair which forms with the given pair a twisted quadrilateral. Calling two such pairs complementary pairs, we have the following result: if we draw two quadrics one through each of two complementary pairs of non-intersecting lines, their residual intersections with S_4 are two sextics of genus one lying on a quartic through D_2.

EXAMPLES OF THE FORMAL ANALYZATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF CERTAIN CLASSES

By E. B. ELLIOTT.

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1. In the *Proceedings of the Fifth International Congress, Cambridge*, 1912 (Vol. 1, p. 344), and elsewhere, I have applied the identity among polynomials

$$1 = \sum_1^m \left\{ \frac{1}{F'(a_r)} \frac{F(\phi)}{\phi - a_r} \right\}, \tag{1}$$

where

$$F(\phi) \equiv (\phi - a_1)(\phi - a_2) \ldots (\phi - a_m), \tag{2}$$

to the analyzation into parts, by direct operations, of functions of various classes, in such ways as to yield results of theoretical interest; and in the *Messenger of Mathematics* (Vol. 45, p. 33) I have found the method useful in connexion with a study of criteria for exact derivatives. An enlargement of scope is now introduced, and in particular (§§ 6, 7) I consider a further analyzation akin to that of the paper in the *Messenger*.

The simple idea is to take with $F(\phi)$ a second polynomial, also without repeated zeros,

$$G(\psi) \equiv (\psi - b_1)(\psi - b_2) \ldots (\psi - b_n), \tag{3}$$

and use the identity among polynomials

$$1 = \sum_{r=1}^{r=m} \sum_{s=1}^{s=n} \left\{ \frac{1}{F'(a_r)\, G'(b_s)} \frac{F(\phi)\, G(\psi)}{(\phi - a_r)(\psi - b_s)} \right\}. \tag{4}$$

Taking for ϕ, ψ any two *commutative* differential operators in any number of variables, we observe that the identity effects the separation of any function u of these variables into mn parts

$$u = \Sigma\Sigma u_{rs} \quad (r = 1, 2, \ldots, m, \ s = 1, 2, \ldots, n), \tag{5}$$

where

$$u_{rs} = \frac{1}{F'(a_r)\, G'(b_s)} \frac{F(\phi)\, G(\psi)}{(\phi - a_r)(\psi - b_s)} u, \tag{6}$$

which is derived from u by direct operations.

2. First suppose that u is a solution of the differential equation

$$F(\phi)\, G(\psi)\, u = f(x, y, z, \ldots). \tag{7}$$

Then any part u_{rs}, obtained as in (6), satisfies a corresponding differential equation

$$(\phi - a_r)(\psi - b_s)\, u_{rs} = \frac{1}{F'(a_r)\, G'(b_s)} f(x, y, z, \ldots). \tag{8}$$

Not only is it true that, when a given solution u of (7) is separated into mn parts by direct operations as above, every part u_{rs} is a solution of the corresponding equation (8), but conversely it is true that when, for every combination of an r chosen from among 1, 2, ..., m and an s chosen from among 1, 2, ..., n, we take any solution u_{rs} whatever of the single corresponding equation (8), and add together the whole mn functions u_{rs}, the sum is a solution of (7). In fact, we have

$$F(\phi)\, G(\psi) \Sigma\Sigma u_{rs} = \sum_{r=1}^{r=m} \sum_{s=1}^{s=n} \left\{ \frac{F(\phi)\, G(\psi)}{(\phi - a_r)(\psi - b_s)} \frac{1}{F'(a_r)\, G'(b_s)} f(x, y, z, \ldots) \right\}$$

$$= f(x, y, z, \ldots),$$

by (4). Consequently the general solution of (7) is the sum of the general solutions of the mn equations (8) taken singly.

In like manner, if we take any number of operators $\phi_1, \phi_2, \phi_3, \ldots$, all commutative with one another, and the same number of polynomials $F_1, F_2, F_3, \ldots$, of degrees $n_1, n_2, n_3, \ldots$, without repeated zeros, the general solution of

$$F_1(\phi_1)\, F_2(\phi_2)\, F_3(\phi_3) \ldots u = f(x, y, z, \ldots), \tag{9}$$

is the sum of the general solutions of the separate $n_1 n_2 n_3 \ldots$ equations

$$(\phi_1 - a_r^{(1)})(\phi_2 - a_s^{(2)})(\phi_3 - a_t^{(3)}) \ldots u_{rst \ldots}$$

$$= \frac{1}{F_1'(a_r^{(1)})\, F_2'(a_s^{(2)})\, F_3'(a_t^{(3)}) \ldots} f(x, y, z, \ldots), \tag{10}$$

with $r = 1, 2, \ldots, n_1$, $s = 1, 2, \ldots, n_2$, $t = 1, 2, \ldots, n_3, \ldots$; and a given solution u of (9) can be written as a sum of $n_1 n_2 n_3 \ldots$ parts, obtained from it by direct operations, which are solutions of the $n_1 n_2 n_3 \ldots$ equations (10) severally.

3. Next let u be a common solution of the two simultaneous equations

$$F(\phi)\, u = f(x, y, z, \ldots), \quad G(\psi)\, u = g(x, y, z, \ldots), \tag{11}$$

where for consistency it is of course necessary that

$$G(\psi)\, f(x, y, z, \ldots) = F(\phi)\, g(x, y, z, \ldots) = h(x, y, z, \ldots), \tag{12}$$

say, so that

$$F(\phi)\, G(\psi)\, u = h(x, y, z, \ldots). \tag{13}$$

The above applies to this, as to (7). But now the mn parts of u, determined by direct operations as in (6), are, by (11), common solutions each of a pair of simpler equations

$$\left.\begin{aligned} (\phi - a_r)\, u_{rs} &= \frac{1}{F'(a_r)\, G'(b_s)}\, \frac{G(\psi)}{\psi - b_s} f(x, y, z, \ldots), \\ (\psi - b_s)\, u_{rs} &= \frac{1}{F'(a_r)\, G'(b_s)}\, \frac{F(\phi)}{\phi - a_r}\, g(x, y, z, \ldots), \end{aligned}\right\} \tag{14}$$

with right-hand sides derived from f and g respectively by direct operations. These right-hand sides are functions $f_s(x, y, z, \ldots)$, $g_r(x, y, z, \ldots)$, such that, by (12),

$$(\psi - b_s)\, f_s(x, y, z, \ldots) = \frac{1}{F'(a_r)\, G'(b_s)}\, h(x, y, z, \ldots) = (\phi - a_r)\, g_r(x, y, z, \ldots). \tag{15}$$

Conversely, if for each of $r = 1, 2, \ldots, m$, and each of $s = 1, 2, \ldots, n$, there is a u_{rs} satisfying the pair of equations (14), then $\Sigma\Sigma u_{rs}$ is a u satisfying the pair (11). This is immediate by use of (4) as in § 2.

4. By way of exemplification, let us obtain facts akin to the known one that a result of n successive integrations $\iiint \ldots f(x)\, dx^n$ can be expressed as a sum of determinate multiples of n terms $x^{r-1} \int x^{n-r} f(x)\, dx$, each involving a single integration.

Consider the differential equation

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\, u = f(x, y), \tag{16}$$

to the form of which can be reduced any equation

$$\left(a \frac{\partial}{\partial \xi} + b \frac{\partial}{\partial \eta} + c\right)^m \left(a' \frac{\partial}{\partial \xi} + b' \frac{\partial}{\partial \eta} + c'\right)^n v = f_1(\xi, \eta) \quad (ab' \neq a'b),$$

by putting $ax + a'y$, $bx + b'y$, for ξ, η, and multiplying by $e^{cx + c'y}$.

The direct method of solution obtains u as a result of $m+n$ successive integrations, m with regard to x, and n with regard to y, there being

arbitrariness to the extent of an additive

$$c_0(x)+c_1(x)y+c_2(x)y^2+\ldots+c_{n-1}(x)y^{n-1}$$

$$+\gamma_0(y)+\gamma_1(y)x+\gamma_2(y)x^2+\ldots+\gamma_{n-1}(y)x^{n-1},$$

where the $m+n$ functions are arbitrary.

But use of § 2 enables us to express the general solution u by means of double integrals, and to write any given solution as a sum of parts, each capable of specification, by means of one double integral, by direct operations.

To see this write (16) as

$$x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}-1\right)\ldots\left(x\frac{\partial}{\partial x}-m+1\right)y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}-1\right)\ldots\left(y\frac{\partial}{\partial y}-n+1\right)u$$

$$=x^m y^n f(x, y),$$

which has the form of (7). The general solution u is the sum of mn parts u_{rs}, each of which, by (8), satisfies the corresponding equation

$$\left(x\frac{\partial}{\partial x}-r+1\right)\left(y\frac{\partial}{\partial y}-s+1\right)u_{rs}=k_{rs}x^m y^n f(x, y),$$

i.e.

$$\frac{\partial^2}{\partial x\partial y}(x^{-r+1}y^{-s+1}u_{rs})=k_{rs}x^{m-r}y^{n-s}f(x, y),$$

where

$$k_{rs}=\frac{(-1)^{m+n-r-s}}{(r-1)!\,(m-r)!\,(s-1)!\,(n-s)!},$$

0! meaning 1. Thus the general solution u of (16) is

$$u=\sum_{r=1}^{r=m}\sum_{s=1}^{s=n}\left\{k_{rs}x^{r-1}y^{s-1}\int^x\int^y x^{m-r}y^{n-s}f(x, y)\,dx\,dy\right\}. \qquad (17)$$

Each double integral has an additive $c(x)+\gamma(y)$ as arbitrary part. The superfluity of the total number of arbitrary functions is only apparent.

The analyzation of any particular given solution u into parts, each effectively a term $k_{rs}x^{r-1}y^{s-1}\int^x\int^y$ as in (17), is effected by direct operations on u as in (6).

5. Consider now the two simultaneous equations

$$\frac{\partial^m u}{\partial x^m}=f(x, y),\quad \frac{\partial^n u}{\partial y^n}=g(x, y), \qquad (18$$

for the consistency of which the condition

$$\frac{\partial^n}{\partial y^n} f(x, y) = \frac{\partial^m}{\partial x^m} g(x, y) = h(x, y), \tag{19}$$

say, is necessary. It will be made clear incidentally that this necessary condition is also sufficient.

The two equations may be written

$$x \frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x} - 1\right) \ldots \left(x \frac{\partial}{\partial x} - m + 1\right) u = x^m f(x, y),$$

$$y \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y} - 1\right) \ldots \left(y \frac{\partial}{\partial y} - n + 1\right) u = y^n g(x, y),$$

and, by § 3, we are told that a common solution u may be expressed as a sum of mn parts u_{rs}, given when u is known by direct operations, of which each satisfies a corresponding pair of equations

$$\left.\begin{aligned}\left(x \frac{\partial}{\partial x} - r + 1\right) u_{rs} &= k_{rs} y \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y} - 1\right) \ldots \left(y \frac{\partial}{\partial y} - s + 2\right) \\ &\qquad \left(y \frac{\partial}{\partial y} - s\right) \ldots \left(y \frac{\partial}{\partial y} - n + 1\right) \{x^m f(x, y)\}, \\ \left(y \frac{\partial}{\partial y} - s + 1\right) u_{rs} &= k_{rs} x \frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x} - 1\right) \ldots \left(x \frac{\partial}{\partial x} - r + 2\right) \\ &\qquad \left(x \frac{\partial}{\partial x} - r\right) \ldots \left(x \frac{\partial}{\partial x} - m + 1\right) \{y^n g(x, y)\},\end{aligned}\right\} \tag{20}$$

where k_{rs} has its previous meaning.

Now, on the condition (19), the pair of equations (20), for any r, s in question, is consistent for the determination of u_{rs}, but for an additive arbitrary multiple of $x^{r-1} y^{s-1}$. For, calling the right hand sides $k_{rs} v_s$ and $k_{rs} w_r$, the pair may be written

$$\frac{\partial}{\partial x}(x^{-r+1} y^{-s+1} u_{rs}) = x^{-r} y^{-s+1} k_{rs} v_s,$$

$$\frac{\partial}{\partial y}(x^{-r+1} y^{-s+1} u_{rs}) = x^{-r+1} y^{-s} k_{rs} w_r$$

and
$$\frac{\partial}{\partial y}(x^{-r}y^{-s+1}v_s) = x^{-r}y^{-s}\left(y\frac{\partial}{\partial y} - s + 1\right)v_s$$
$$= x^{m-r}y^{n-s}\frac{\partial^n}{\partial y^n}f(x, y)$$
$$= x^{m-r}y^{n-s}\frac{\partial^m}{\partial x^m}g(x, y) \quad [\text{by (19)}]$$
$$= \frac{\partial}{\partial x}(x^{-r+1}y^{-s}w_r),$$

in like manner, so that

$$x^{-r}y^{-s+1}v_s dx + x^{-r+1}y^{-s}w_r dy$$

is an exact derivative, and

$$u_{rs} = k_{rs}x^{r-1}y^{s-1}\int x^{-r}y^{-s}(yv_s dx + xw_r dy). \tag{21}$$

These terms being thus existent and determined, on the condition, so is their sum; and, by § 3, the sum $\Sigma\Sigma u_{rs}$, for

$$r = 1, 2, \ldots, m, \qquad s = 1, 2, \ldots, n,$$

is the general solution of the pair of equations (18).

6. We proceed to apply § 3 to a more complex problem of analyzation in continuation of a former paper.*

Take
$$u \equiv f(x, y, z, z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, \ldots), \tag{22}$$

any function of two independent variables x, y, a dependent z, and the successive derivatives z_{rs} of z with regard to x and y, which is (1) homogeneous of degree i ($\neq 0$) in z and the derivatives, and (2) rational and integral in the derivatives, though not necessarily in z.

Denote by D and D' the operators of x- and y-partial differentiation,

$$D \equiv \frac{\partial}{\partial x} + z_{10}\frac{\partial}{\partial z} + z_{20}\frac{\partial}{\partial z_{10}} + z_{11}\frac{\partial}{\partial z_{01}} + \ldots \equiv \frac{\partial}{\partial x} + \Sigma z_{r+1,s}\frac{\partial}{\partial z_{rs}},$$

$$D' \equiv \frac{\partial}{\partial y} + z_{01}\frac{\partial}{\partial z} + z_{11}\frac{\partial}{\partial z_{10}} + z_{02}\frac{\partial}{\partial z_{01}} + \ldots \equiv \frac{\partial}{\partial y} + \Sigma z_{r,s+1}\frac{\partial}{\partial z_{rs}}$$

$(r, s = 0, 1, 2, 3, \ldots),$

* "A Set of Criteria for Exact Derivatives," *Messenger of Mathematics*, Vol. 45, pp. 33–39.

and also take the two operators

$$\omega = \Sigma(r+1)\, z_{rs} \frac{\partial}{\partial z_{r+1,s}}, \qquad \omega' = \Sigma(s+1)\, z_{rs} \frac{\partial}{\partial z_{r,s+1}}.$$

We observe that, on whatever we operate,

$$\left.\begin{aligned} DD'-D'D &= 0, \quad & \omega\omega'-\omega'\omega &= 0, \\ D\omega'-\omega'D &= 0, \quad & D'\omega-\omega D' &= 0, \end{aligned}\right\} \tag{23}$$

so that $$D\omega D'\omega' = DD'\omega\omega' = D'D\omega'\omega = D'\omega' D\omega,$$

and consequently $D\omega$, $D'\omega'$ are commutative operators. We will use them for the ϕ, ψ of §§ 1–3.

We have also

$$\begin{aligned} D\omega-\omega D = D'\omega'-\omega'D' &= -\Sigma z_{rs} \frac{\partial}{\partial z_{rs}} \\ &= -i, \end{aligned} \tag{23a}$$

when, as now, the operations are on a homogeneous function of degree i.

By use of this we find, as in *Messenger, loc. cit.*, p. 35, that for every n

$$\left.\begin{aligned} D^{n+1}\omega^{n+1} &= D\omega(D\omega-i)(D\omega-2i)\ldots(D\omega-ni), \\ \text{and} \quad D'^{n+1}\omega'^{n+1} &= D'\omega'(D'\omega'-i)(D'\omega'-2i)\ldots(D'\omega'-ni). \end{aligned}\right\} \tag{24}$$

Now let w be the greatest sum of first suffixes in any term of u, and w' the greatest sum of second suffixes in any term. Then

$$\omega^{w+1}u = 0, \qquad \omega'^{w'+1}u = 0,$$

so that $$D^{w+1}\omega^{w+1}u = 0, \qquad D'^{w'+1}\omega'^{w'+1}u = 0. \tag{25}$$

Thus u is a common solution of the two equations

$$\left.\begin{aligned} D\omega(D\omega-i)(D\omega-2i)\ldots(D\omega-wi)\,u &= 0, \\ D'\omega'(D'\omega'-i)(D'\omega'-2i)\ldots(D'\omega'-w'i)\,u &= 0. \end{aligned}\right\} \tag{26}$$

Take these for the two equations (11) of § 3. The right-hand sides f and g are zero. We learn that

$$u = \Sigma\Sigma u_{rs} \quad (r = 0, 1, 2, \ldots, w\,;\ s = 0, 1, 2, \ldots, w'), \tag{27}$$

where every u_{rs} satisfies the corresponding pair of equations

$$(D\omega-ri)\,u_{rs} = 0, \qquad (D'\omega'-si)\,u_{rs} = 0, \tag{28}$$

and is given by direct operations on u as

$$\begin{aligned} u_{rs} &= k_{rs} D\omega(D\omega - i) \dots (D\omega - \overline{r-1}\, i)(D\omega - \overline{r+1}\, i) \dots (D\omega - wi) \\ &\qquad D'\omega'(D'\omega' - i) \dots (D'\omega' - \overline{s-1}\, i)(D'\omega' - \overline{s+1}\, i) \dots (D'\omega' - w'i)\, u \\ &= k_{rs} D^r \omega^r (D\omega - \overline{r+1}\, i) \dots (D\omega - wi)\, D'^s \omega'^s (D'\omega' - \overline{s+1}\, i) \dots (D'\omega' - w'i)\, u \\ &= k_{rs} D^r D'^s \omega^r \omega'^s (D\omega - \overline{r+1}\, i) \dots (D\omega - wi)(D'\omega' - \overline{s+1}\, i) \dots (D'\omega' - w'i) u, \end{aligned} \tag{29}$$

where $$k_{rs} = \frac{(-1)^{w+w'-r-s}}{r!\,(w-r)!\,s!\,(w'-s)!}.$$

Thus u_{rs} is an (r, s)-th derivative $D^r D'^s v_{rs}$.

Notice that if W, the greatest sum of all suffixes (first and second) in any single term of u, is less than $w+w'$, those parts $D^r D'^s v_{rs}$ in which $r+s > W$ will vanish. For, with such an r, s, $\omega^r \omega'^s u = 0$.

We have then expressed our u by direct operation as a sum of $(w+1)(w'+1)$, or fewer, parts u_{rs}

$$(r = 0, 1, 2, \dots, w\,;\ s = 0, 1, 2, \dots, w'\,;\ r+s \not> W),$$

of which every part u_{rs} is an r-th derivative with regard to x, and an s-th with regard to y. It will be seen presently that no u_{rs} is a higher derivative than here stated with regard to either variable.

The whole u will be an (r, s)-th derivative if every $u_{\rho\sigma}$ with either $\rho < r$ or $\sigma < s$ vanishes. And this, by (29), will be the case if u satisfies both of

$$\left.\begin{aligned} (D\omega - ri)(D\omega - \overline{r+1}\, i) \dots (D\omega - wi)\, u = 0, \\ (D'\omega' - si)(D'\omega' - \overline{s+1}\, i) \dots (D'\omega' - w'i)\, u = 0. \end{aligned}\right\} \tag{30}$$

Moreover, conversely, every u, of characteristics w, w', which is an (r, s)-th derivative satisfies this pair of equations. For, using (23a), we find, as in *Messenger, loc. cit.*, that

$$\begin{aligned} &(D\omega - ri)(D\omega - \overline{r+1}\, i) \dots (D\omega - wi) D^r D'^s v \\ &\qquad = D^r D\omega (D\omega - i) \dots (D\omega - \overline{w-r}\, i)\, D'^s v \\ &\qquad = D^r D'^s\, D^{w-r+1} \omega^{w-r+1} v, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &(D'\omega' - si)(D'\omega' - \overline{s+1}\, i) \dots (D'\omega' - w'i)\, D^r D'^s v \\ &\qquad = D^r D'^s\, D'^{w'-s+1} \omega'^{w'-s+1} v, \end{aligned} \tag{33}$$

where the right-hand members both vanish by (25), because the maximum

weight characteristics w, w' of $u \equiv D^r D'^s v$ exceed by r, s respectively those of v.

The statement of the last paragraph but two, that the part u_{rs} of any chosen u, obtained from u in (29), and there proved to be an (r, s)-th derivative, is not either an $(r+1, s)$-th or an $(r, s+1)$-th derivative, can now be proved. By (28) it satisfies

$$(D\omega - ri)\, u_{rs} = 0.$$

If it were an $(r+1, s)$-th derivative, we have now seen that it would also have to satisfy

$$(D\omega - \overline{r+1}\, i)(D\omega - \overline{r+2}\, i) \ldots (D\omega - wi) u_{rs} = 0,$$

and we should have

$$(r-r-1)(r-r-2) \ldots (r-w)\, i^{w-r}\, u_{rs} = 0,$$

which, as we have excluded the cases of degree $i = 0$, is not true unless $u_{rs} = 0$. Similarly, unless it is zero, u_{rs} is not an $(r, s+1)$-th derivative.

This, in fact, proves that any u satisfying

$$(D\omega - ri)\, u = 0, \qquad (D'\omega' - si) u = 0, \tag{34}$$

is an (r, s)-th but not either an $(r+1, s)$-th or $(r, s+1)$-th derivative. The two conditions are *sufficient* to ensure that this is the case, but not *necessary*, except for $r = w$, $s = w'$. The necessary and sufficient conditions for u to be a $D^r D' v$ have been seen to be the pair (30). The necessary and sufficient additional conditions that it be not also either a $D^{r+1} D'^s v'$ or a $D^r D'^{s+1} v'$ are that it do not satisfy either the equation obtained by removing the left-hand operative factor $Dw - ri$ from the first of the pair (30), or that obtained by removing $D'\omega' - s'i$ from the second.

As above, the satisfaction of the pair (34) by u secures all four conditions; but the converse is not true. An (r, s)-th but not higher derivative remains (r, s)-th but not higher if we add to or take from it any higher derivative.

7. In the above there has been a single dependent variable z. We may instead take any number of dependent variables $z^{(1)}$, $z^{(2)}$, $z^{(3)}$, ...

If u is a function of x, y and the various dependent variables and their derivatives, which is (1) homogeneous of non-zero degree i in the dependent variables and derivatives collectively, and (2) rational and integral in all the derivatives, if further w is the greatest sum of first

suffixes in any term of u, and w' the greatest sum of second suffixes in any term, and if

$$D \equiv \frac{\partial}{\partial x} + \Sigma z^{(1)}_{r+1,\,s} \frac{\partial}{\partial z^{(1)}_{rs}} + \Sigma z^{(2)}_{r+1,\,s} \frac{\partial}{\partial z^{(2)}_{rs}} + \ldots,$$

$$D' \equiv \frac{\partial}{\partial y} + \Sigma z^{(1)}_{r,\,s+1} \frac{\partial}{\partial z^{(1)}_{rs}} + \Sigma z^{(2)}_{r,\,s+1} \frac{\partial}{\partial z^{(2)}_{rs}} + \ldots,$$

$$\omega \equiv \Sigma (r+1)\, z^{(1)}_{rs} \frac{\partial}{\partial z^{(1)}_{r+1,\,s}} + \Sigma (r+1)\, z^{(2)}_{rs} \frac{\partial}{\partial z^{(2)}_{r+1,\,s}} + \ldots,$$

$$\omega' \equiv \Sigma (s+1) z^{(1)}_{rs} \frac{\partial}{\partial z^{(1)}_{r,\,s+1}} + \Sigma (s+1)\, z^{(2)}_{rs} \frac{\partial}{\partial z^{(2)}_{r,\,s+1}} + \ldots,$$

the summations covering $r, s = 0, 1, 2, \ldots, \infty$, everything as expressed still applies without alteration.

Moreover, if u is a function of x, y and the various dependent variables and their derivatives, which is (1) homogeneous of non-zero degree i in a single one $z^{(1)}$ of the dependent variables and its derivatives, and (2) rational and integral in the derivatives of that single $z^{(1)}$, and if w is the greatest sum of first suffixes of the single $z^{(1)}$ in any term and w' the greatest sum of its second suffixes in any term, all that has been given still holds verbally when we mean by D, D' the complete operators of partial differentiation above, but by ω, ω' the parts of the above ω, ω' which refer to $z^{(1)}$ only, *i.e.*

$$\omega = \Sigma (r+1)\, z^{(1)}_{rs} \frac{\partial}{\partial z^{(1)}_{r+1,\,s}}, \qquad \omega' = \Sigma (s+1)\, z^{(1)}_{rs} \frac{\partial}{\partial z^{(1)}_{r,\,s+1}}.$$

NOTE ON A DIOPHANTINE APPROXIMATION

By J. H. GRACE.

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IT has long been known that, if a is an irrational number and b any number, integers x and y can be found so that

$$ay-x-b \tag{I}$$

is arbitrarily small.

Tchebychef proved that there is an infinite number of integer y's such that

$$|ay-x-b| < \frac{1}{2|y|}, \tag{II}$$

Hermite that the same is true if $\frac{1}{2}$ is replaced by the smaller number $\sqrt{\frac{2}{27}}$, and Minkowski* that

$$|ay-x-b| < \frac{1}{4|y|} \tag{III}$$

holds for an infinite number of integer values of y.

It is explicitly stated by Minkowski that for the truth of (III) b must not be an integer, and, in fact, it had previously been practically established by Markoff and Hurwitz† that, when b is an integer, the strongest general inequality of type (II) is

$$|ay-x-b| < \frac{1}{\sqrt{5}|y|}$$

Inasmuch as $\sqrt{\frac{2}{27}}$ is less than $\sqrt{\frac{1}{5}}$, and Hermite's argument does not depend on b being non-integer, that argument needs modification before

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* See Minkowski, *Werke*, Vol. 1, p. 320. Hermite's paper occurs in his *Œuvres*, Vol. 3, and in *Crelle's Journal* for 1879.

† Hurwitz, *Math. Ann.*, Vol. 39, p. 279. Cf. Klein's lithographed lectures on the Theory of Numbers.

it can be accepted. It seems probable that once the method is perfected it can be extended to two or more linear forms of the type (I).*

The object of this note, however, is to prove that Minkowski's result is final, *i.e.* that, if $k < \frac{1}{4}$, it is possible to choose a and b so that there is not an infinite number of integers y for which

$$|ay-x-b| < \frac{k}{|y|}.$$

The argument is quite simple, but it does not establish the result of Minkowski, only that no stronger result can be universally true. I take $b = \frac{1}{2}$, and remark that if,

$$|ay-x-\tfrac{1}{2}| < \frac{k}{|y|},$$

for a negative value of y, the same is true for the corresponding positive value, though not for the same value of x.

Thus, if the result is not final, we should have

$$|ay-x-\tfrac{1}{2}| < \frac{k}{y}$$

for an infinite number of positive values of y and $k < \frac{1}{4}$. That is to say

$$\left| a - \frac{2x+1}{2y} \right| < \frac{k}{y^2},$$

or

$$\left| a - \frac{2x+1}{2y} \right| < \frac{h}{(2y)^2} \quad (h < 1).$$

There would thus be an infinite number of approximations ξ/η to a in which ξ is odd, η even, and

$$\left| a - \frac{\xi}{\eta} \right| < \frac{h}{\eta^2} \quad (h < 1). \tag{IV}$$

It may be assumed that ξ and η are co-prime, for, if they were not, we could deduce a similar inequality in which they were, with a smaller value of h.

Now any approximation satisfying the last named condition must (as can be easily proved) be either a principal, or an auxiliary, convergent of the continued fraction for a. I first arrange that the denominators of all the principal convergents are odd, so that ξ/η is none of these.

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* And so to establish a theorem, the truth of which is conjectured by Hardy and Littlewood, *Acta Math.*, Vol. 37, p. 178, under Theorem 1 . 341.

This is secured by writing

$$a = \frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_n} + \dots = (a_1, a_2, \dots, a_n, \dots),$$

in which a_1 is odd and the other a's all even; the approximation ξ/η obtained above must then be an auxiliary convergent.

Suppose that p_n/q_n is the convergent corresponding to a_n, and that p/q is an auxiliary convergent between p_n/q_n and p_{n+1}/q_{n+1}; then we have

$$\frac{p}{q} = \frac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}},$$

where α is an integer less than a_{n+1}. A simple calculation gives

$$\left| \frac{p}{q} - a \right| = \frac{1}{\lambda q^2},$$

where

$$\lambda = (\alpha, a_n, a_{n-1}, \dots) + (\beta, a_{n+2}, a_{n+3}, \dots), \qquad (V)$$

and

$$\alpha + \beta = a_{n+1}.$$

If p/q satisfies the condition (IV) we must have

$$\lambda > \frac{1}{h} > 1:$$

hence either $\alpha = 1$ or $\beta = 1$. Taking the two cases we find

$$\lambda < 1 + \frac{1}{a_{n+1} - 1}$$

in each.

It is now quite clear that, if the a's steadily increase without limit, there will not be an infinite number of such λ's greater than $1/h$, if h is less than unity. This is the result required. Briefly, if we take $b = \frac{1}{2}$ and

$$a = (a_1, a_2, \dots, a_n, \dots),$$

when a_1 is odd, the other a's even and steadily increasing, there is not an infinite number of values of y for which

$$|ay - x - b| < \frac{1}{k|y|} \quad \left(k < \frac{1}{4}\right).$$

It may be noted that, even in this case, there is an infinite number

of values of y for which

$$|ay-x-b| < \frac{1}{4|y|}.$$

In fact, with either $\alpha = 1$ or $\beta = 1$ above, p is odd and q is even, and from (V)

$$\lambda > \frac{1}{1+1/a_n} + \frac{1}{a_{n+1}} > 1 - \frac{1}{a_n} + \frac{1}{a_{n+1}},$$

and

$$\lambda > 1 - \frac{1}{a_{n+2}} + \frac{1}{a_{n+1}},$$

in the two cases. With our conditions the second value of λ exceeds unity; so, taking a p/q for which $\beta = 1$, we get the result, which is, of course, in accordance with the general one of Minkowski.

ON THE CONTINUATION OF THE HYPERGEOMETRIC SERIES

By Prof. M. J. M. HILL.

[Received September 11th, 1917.—Read November 1st, 1917.]

1. The idea of the continuation of a series is algebraic in origin, so that, from a theoretical point of view, the continuation of a series $P(z)$ should be obtainable by expanding it in powers of $z-z_1$ by Taylor's theorem, and then expanding the series again in powers of $z-z_2$, and so on. But this process generally presents very great difficulty, and the continuations are usually obtained by contour integration or the known results as to the theory of the solution of differential equations. It would seem, however, to be of interest to obtain the continuation of a few series by algebraic processes only.

2. In the year 1902 I obtained by *ordinary algebraic expansion* the continuations of two series which have one singular point, viz., the binomial and logarithmic series; and of two series which have two singular points, viz., the series for arc tan x and arc sin x (*Proc. London Math. Soc.*, Ser. 1, Vol. xxxv, pp. 388–416).

The object of this paper is to apply the same method to the case next in order of complexity, viz. to the hypergeometric series, which has three singular points. The analytical difficulties in carrying through the work are different in kind to those which present themselves in the series mentioned above.

3. It is sufficient to prove the equation

$$\begin{aligned}&\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x)\\ =\ &\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma+1)\,F(\alpha,\ \beta,\ \gamma,\ x)\\ &+\Big[\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-1)\,\Gamma(\alpha+\beta-\gamma+1)\,x^{1-\gamma}\\ &\qquad\qquad\times F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x)\Big],\end{aligned}\qquad (\mathrm{I})$$

because the other relations of the same kind can be obtained from it by

the aid of Euler's identities and appropriate transformations of the variable and the parameters. Purely algebraic proofs of Euler's identities will be found in the *Messenger of Mathematics*, New Series, No. 524, December 1914. I have been unable to find Euler's or any other purely algebraic proof of Euler's identities.

4. The six fundamental solutions of the differential equation satisfied by the hypergeometric series can be taken as

(i) $F(\alpha, \beta, \gamma, x)$,

(ii) $x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$,

(iii) $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$,

(iv) $(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$,

(v) $x^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, x^{-1})$,

(vi) $x^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1, x^{-1})$.

These are all finite at the singular points in their respective domains if the real parts of α, β, $1-\gamma$ and $\gamma-\alpha-\beta$ are positive.

The demonstration is greatly simplified if we suppose this to be the case, and if moreover we suppose the parameters to be real. This makes α, β, γ and $\gamma-\alpha-\beta$ all positive proper fractions. It will be seen that the coefficients of every power of x in each of the six fundamental solutions are in this case positive.

5. The following notation will be used in this paper.

The product $\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1)$ will be denoted by α_r.

And
$$G\begin{pmatrix}\alpha, \beta, \\ \gamma, \delta, \end{pmatrix} s$$
will denote the sum of the series

$$1+\sum_{p=1}^{s}(\alpha_p\beta_p)/(\gamma_p\delta_p). \qquad \text{(II)}$$

Then $G\begin{pmatrix}\alpha, \beta, \\ 1, \gamma, \end{pmatrix}\infty$ is the series usually denoted by $F(\alpha, \beta, \gamma, 1)$, and we have

$$G\begin{pmatrix}\alpha, \beta, \\ 1, \gamma, \end{pmatrix}\infty = \{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)\}/\{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\}, \qquad \text{(III)}$$

provided that the real part of $\gamma-\alpha-\beta$ is positive.

6. In the *Proceedings of the London Mathematical Society*, 1907, pp. 335–338, it is shown by purely algebraic methods that

$$G\begin{pmatrix} a, & \beta, \\ 1, & \gamma, \end{pmatrix} s = \{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)\}/\{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\}$$
$$-\{(\alpha_{s+1}\beta_{s+1})/(s!\,\gamma_{s+1})\}(\gamma-\alpha-\beta)^{-1}\,G\begin{pmatrix} \gamma-\alpha, & \gamma-\beta, \\ \gamma-\alpha-\beta+1, & \gamma+s+1, \end{pmatrix} \infty. \quad \text{(IV)}$$

7. In the *Quarterly Journal of Pure and Applied Mathematics*, 1910, Vol. 41, pp. 128–135, there is an algebraic proof by Mr. Whipple and the writer of this paper of the theorem that

$$(\delta-1)^{-1}\,G\begin{pmatrix} a, & \beta, \\ \gamma, & \delta, \end{pmatrix} \infty$$
$$= (\gamma+\delta-\alpha-\beta-1)^{-1}\,G\begin{pmatrix} \gamma-\alpha, & \gamma-\beta, \\ \gamma, & \gamma+\delta-\alpha-\beta, \end{pmatrix} \infty, \quad \text{(V)}$$

provided that the real parts of δ and of $\gamma+\delta-\alpha-\beta$ are each greater than unity.

8. The theorem that

$$(a+b)_n = \sum_{r=0}^{n} {}_nC_r\,a_{n-r}\,b_r \quad \text{(VI)}$$

can be deduced from Vandermonde's theorem or proved independently as follows.

The theorem is obviously true for $n = 2$ and 3.

Assuming it to be true for any integral value of n, multiply each side by $(a+b+n)$.

The left-hand side becomes $(a+b)_{n+1}$.

And since
$$(a+b+n)\,a_{n-r}\,b_r$$
$$= (a+n-r)\,a_{n-r}\,b_r+(b-r)\,a_{n-r}\,b_r$$
$$= a_{n-r+1}\,b_r+a_{n-r}\,b_{r+1},$$

the right-hand side becomes

$$\sum_{r=0}^{n} {}_nC_r(a_{n-r+1}\,b_r+a_{n-r}\,b_{r+1})$$
$$= \sum_{r=0}^{n} ({}_nC_r+{}_nC_{r-1})\,a_{n+1-r}\,b_r+{}_nC_n\,a_0\,b_{n+1}$$
$$= \sum_{r=0}^{n+1} {}_{n+1}C_r\,(a_{n+1-r}\,b_r);$$

therefore $$(a+b)_{n+1} = \sum_{r=0}^{n+1} {}_{n+1}C_r \, a_{n+1-r} b_r,$$

and now the theorem follows by induction.

9. The series $$F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x),$$

and $$x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x),$$

cannot be expressed in series of positive integral powers of x.

Consider therefore the first $(n+1)$ terms of the first of these series.

Expanding each term in powers of x, it will be found that the coefficient of x^r is

$$[(-1)^r \alpha_r \beta_r / \{r! \, (\alpha+\beta-\gamma+1)_r\}]$$

$$\times G\begin{pmatrix} \alpha+r, & \beta+r, \\ 1, & \alpha+\beta-\gamma+1+r, \end{pmatrix} n-r\Big). \quad \text{(VII)}$$

Applying equation (IV) to transform the series G in (VII), the whole expression (VII) can be replaced by another consisting of two parts.

The first of these parts is

$$[(-1)\, r\alpha_r \beta_r / \{r! \, (\alpha+\beta-\gamma+1)_r\}]$$

$$\times \Gamma(\alpha+\beta-\gamma+1+r) \, \Gamma(1-\gamma-r) / \{\Gamma(\alpha-\gamma+1) \, \Gamma(\beta-\gamma+1)\},$$

which reduces to

$$[\alpha_r \beta_r / \{r! \, \gamma_r\}] \, \Gamma(1-\gamma) \, \Gamma(\alpha+\beta-\gamma+1) / \{\Gamma(\alpha-\gamma+1) \, \Gamma(\beta-\gamma+1)\}, \quad \text{(VIII)}$$

and thus accounts for the coefficient of x^r in the term containing $F(\alpha, \beta, \gamma, x)$ in (I), after that equation has been divided by the coefficient of $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ on the left.

10. The second part of the coefficient of x^r is

$$[(-1)^{r+1} \alpha_r \beta_r / \{r! \, (\alpha+\beta-\gamma+1)_r\}]$$

$$\times [(\alpha+r)_{n-r+1} \, (\beta+r)_{n-r+1} / \{(n-r)! \, (\alpha+\beta-\gamma+1+r)_{n-r+1}\}]$$

$$\times (1-\gamma-r)^{-1} \, G\begin{pmatrix} \alpha-\gamma+1, & \beta-\gamma+1, \\ 2-\gamma-r, & \alpha+\beta-\gamma+n+2, \end{pmatrix} \infty\Big). \quad \text{(IX)}$$

The series in (IX) satisfies the condition of convergence.

It is necessary in what follows to separate it into two parts, taking the first $n+1$ terms in the first part, and the remaining terms in the second part.

The expression (IX) can be reduced to the form

$$[a_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}\,(\gamma-1)_{n+1}\}] \tag{X}$$

$$\times[(-1)^{r+1}\,(\gamma-1)_{n+1}/\{r!\,(n-r)!\}] \tag{XI}$$

$$\times(1-\gamma-r)^{-1}\,G\begin{pmatrix}\alpha-\gamma+1,\ \beta-\gamma+1,\\ 2-\gamma-r,\ \alpha+\beta-\gamma+n+2,\end{pmatrix}\infty\Big). \tag{XII}$$

The $(p+1)$-th term in the series (XII) is

$$(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}. \tag{XIII}$$

Now it can be shown by induction, or by the method of partial fractions that

$$p!/a_{p+1} = \sum_{s=0}^{p}(-1)^s\,{}_pC_s/(a+s);$$

therefore $$p!/(1-\gamma-r)_{p+1} = \sum_{s=0}^{p}(-1)^s\,{}_pC_s/(1-\gamma-r+s$$

therefore

$$(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}$$
$$=[(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,p!\}]$$
$$\times\Big[\sum_{s=0}^{p}(-1)^s\,{}_pC_s/(1-\gamma-r+s)\Big].$$

Consequently the terms in the series (XII), which give rise to terms containing $(1-\gamma-r+k)^{-1}$, are those for which $p \geqslant k$.

As, however, we consider in the first instance only the first $n+1$ terms of the series (IX), we must take $p = k,\ k+1,\ \ldots,\ n$. Also $s = k$.

If now we write t_p for

$$(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{p!\,(\alpha+\beta-\gamma+n+2)_p\},$$

then the terms obtainable from the first $(n+1)$ terms in (XII) containing

$(1-\gamma-r+k)^{-1}$ are

$$(-1)^k(1-\gamma-r+k)^{-1}({}_kC_\kappa t_\kappa+{}_{\kappa+1}C_\kappa t_{\kappa+1}+{}_{\kappa+2}C_\kappa t_{\kappa+2}+\ldots+{}_nC_\kappa t_n)$$

$$=\left\{\begin{array}{l}(-1)^k(1-\gamma-r+k)^{-1}(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(\alpha+\beta-\gamma+n+2)_k\}\\ \qquad\times G\left(\begin{matrix}\alpha-\gamma+1+k,\ \beta-\gamma+1+k,\\ 1,\ \alpha+\beta-\gamma+n+2+k,\end{matrix}\ n-k\right)\end{array}\right\}. \tag{XIV}$$

Hence the portion of the second part of the coefficient of x^r which contains $(1-\gamma-r+k)^{-1}$ is the whole expression (XIV) multiplied by the product of the expressions (X) and (XI).

Now $\quad(\gamma-1)_{n+1}=(-1)^k(\gamma-k-1)_{n+1}(\gamma-k+n)_k/(2-\gamma)_k.$

Making use of this, the portion of the second part of the coefficient of x^r which contains $(1-\gamma-r+k)^{-1}$ is

$$\begin{aligned}&\alpha_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}(\gamma-1)_{n+1}\}\\ &\times(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}\\ &\times(\gamma-k+n)_k/(\alpha+\beta-\gamma+n+2)_k\\ &\times(\gamma-k-1)_{n+1}/(1-\gamma-r+k)\\ &\times(-1)^{r+1}/\{r!\,(n-r)!\}\\ &\times G\left(\begin{matrix}\alpha-\gamma+1+k,\ \beta-\gamma+1+k,\\ 1,\ \alpha+\beta-\gamma+n+2+k,\end{matrix}\ n-k\right).\end{aligned} \tag{XV}$$

If the series G in (XV) were continued to infinity, the condition for convergence would be $k<n+\gamma$.

By hypothesis γ is a positive proper fraction and the greatest value of k is n. So the series continued to infinity is convergent.

Let us suppose *for the moment* that we replace the series G in (XV) by its sum to infinity. Since all the coefficients in G are positive, this amounts to an addition to the value of G. We shall investigate the effect of this addition at a later stage.

The sum of the series G in (XV) supposed extended to infinity is

$$\Gamma(\alpha+\beta-\gamma+n+2+k)\,\Gamma(n+\gamma-k)/\{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\}.$$

This is equal to

$$\begin{aligned}&\Gamma(\alpha+\beta-\gamma+n+2)\,\Gamma(n+\gamma)/\{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\}\\ &\times(\alpha+\beta-\gamma+n+2)_k/(n+\gamma-k)_k.\end{aligned}$$

Supposing then G in (XV) to be extended to infinity, (XV) is replaced by

$$\begin{aligned}
&\alpha_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}(\gamma-1)_{n+1}\}\\
&\times\Gamma(\alpha+\beta-\gamma+n+2)\,\Gamma(n+\gamma)/\{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\}\\
&\times(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}\\
&\times(-1)^r(\gamma-k-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-k-1+r)\}\\
=\;&\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)/\{\Gamma(\alpha)\,\Gamma(\beta)\}\\
&\times(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}\\
&\times(-1)^r(\gamma-k-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-k-1+r)\}. \qquad \text{(XVI)}
\end{aligned}$$

The expression (XVI) is that portion of the coefficient of x^r which contains $(\gamma-k-1+r)^{-1}$.

The next step is to sum the product of x^r and the expression (XVI) from $r=0$ to $r=n$.

It is therefore sufficient to calculate

$$\sum_{r=0}^{n}(-1)^r x^r(\gamma-k-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-k-1+r)\}. \qquad \text{(XVII)}$$

11. To determine its meaning, let us expand $x^{1-\gamma}$ in powers of $1-x$, it being assumed that $|1-x|<1$.

The first $(n+1)$ terms of the expansion are

$$1+(\gamma-1)(1-x)+(\gamma-1)_2(1-x)^2/2!+\ldots+(\gamma-1)_n(1-x)^n/n!.$$

Arranging them in a series of ascending powers of x we get

$$\sum_{r=0}^{n}(-1)^r x^r(\gamma-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-1+r)\}.$$

From this the expression (XVII) can be deduced by changing γ into $\gamma-k$.

Hence the expression (XVII) is seen to be the equivalent of the first $n+1$ terms of the expansion of

$$x^{1-\gamma+k},$$

in a series of powers of $1-x$.

To see whether we can replace the expression (XVII) by $x^{1-\gamma+k}$, we must investigate the value of the terms following the $(n+1)$-th. We

therefore require a superior limit for

$$|(\gamma-k-1)_{n+1}(1-x)^{n+1}/(n+1)!+(\gamma-k-1)_{n+2}(1-x)^{n+2}/(n+2)!+\ldots \text{to } \infty|$$
$$=|(\gamma-k-1)_{n+1}(1-x)^{n+1}/(n+1)!|$$
$$\times\left|1+\frac{\gamma-k+n}{n+2}(1-x)+\frac{(\gamma-k+n)_2}{(n+2)_2}(1-x)^2+\ldots\right|.$$

Now, since $k<n+\gamma$, the coefficients of the powers of $1-x$ are all less than 1, and therefore the series last written is less than

$$|1+|1-x|+|1-x|^2+\ldots|;$$

and therefore less than $1/\{1-|1-x|\}$.

Moreover, as n increases,

$$|(\gamma-k-1)_{n+1}(1-x)^{n+1}/(n+1)!|$$

tends to zero.

Hence, provided that $k<n+\gamma$, we can replace the expression (XVII) by $x^{1-\gamma+k}$.

Hence the second part of the coefficient of x^r, if summed from $r=0$ to n, contains the term

$$\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)/\{\Gamma(\alpha)\,\Gamma(\beta)\}$$
$$\times(\alpha-\gamma+1)_k\,(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}$$
$$\times x^{1-\gamma-k},$$

provided that it is admissible to replace the series G in (XV) by its sum to infinity.

This gives the coefficient of $x^{1-\gamma+k}$ in the term containing

$$x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x)$$

in (I), after that equation has been divided by the coefficient of

$$F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x).$$

12. There remain still outstanding two points.

(i) What is the effect on the result of the addition to the series G in (XV) of the terms from the $(n-k+1)$-th to infinity?

(ii) What is the effect of neglecting the terms in the series G in (XII) from the $(n+1)$-th to infinity?

13. As regards the first point it is clear that if the number k (which does not exceed n) tends to infinity with n, then the effect of replacing

$$G\left(\begin{matrix}\alpha-\gamma+1+k,\ \beta-\gamma+1+k,\\ 1,\ \alpha+\beta-\gamma+n+2+k,\end{matrix}\ n-k\right)$$

by its sum to infinity may add largely to the value of the series G in (XV); but the total arrived at, viz.

$$\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)/\{\Gamma(\alpha)\,\Gamma(\beta)\}$$
$$\times(\alpha-\gamma+1)_k\,(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}$$
$$\times x^{1-\gamma+k},$$

is small; and therefore the effect on the whole series to be calculated is small, and by sufficiently increasing k can be made as small as we please. Since n can be made as large as we please, it is possible to make k as large as we please.

14. If, however, k does not tend to infinity with n, it is necessary to show that the sum added to the G in (XV) is small when n is large.

Now the ratio of the sum of the terms added to the total arrived at can be obtained from equation (IV) by dividing the second term on its right-hand side by its first term, and then replacing α, β, γ, s by $\alpha-\gamma+1+k$, $\beta-\gamma+1+k$, $\alpha+\beta-\gamma+n+2+k$, $n-k$ respectively.

This ratio is

$$\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)/\{\Gamma(\alpha+\beta-\gamma+2+n+k)\,\Gamma(n+\gamma-k)\}$$
$$\times\left[\frac{(\alpha-\gamma+1+k)_{n-k+1}\,(\beta-\gamma+1+k)_{n-k+1}}{\{(n-k)!\,(\alpha+\beta-\gamma+2+n+k)_{n-k+1}\}}\right]$$
$$\times(\gamma+n-k)^{-1}\,G\left(\begin{matrix}\alpha+n+1,\ \beta+n+1,\\ \gamma+n-k+1,\ \alpha+\beta-\gamma+3+2n,\end{matrix}\ \infty\right).$$

The series G satisfies the condition of convergence.

From (V) it follows that

$$(\alpha+\beta-\gamma+2n+2)^{-1}\,G\left(\begin{matrix}\alpha+n+1,\ \beta+n+1,\\ \gamma+n-k+1,\ \alpha+\beta-\gamma+2n+3,\end{matrix}\ \infty\right)$$
$$=(n-k+1)^{-1}\,G\left(\begin{matrix}\gamma-\alpha-k,\ \gamma-\beta-k,\\ n+\gamma-k+1,\ n-k+2,\end{matrix}\ \infty\right).$$

Hence the ratio under discussion is

$$\Gamma(a+n+1)\,\Gamma(\beta+n+1)/\{\Gamma(a+\beta-\gamma+2+n+k)\,\Gamma(n+\gamma-k)\}$$

$$\times\left[\frac{(a-\gamma+k+1)_{n-k+1}\,(\beta-\gamma+k+1)_{n-k+1}}{\{(n-k)!\,(a+\beta-\gamma+n+2+k)_{n-k+1}\}}\right]$$

$$\times(a+\beta-\gamma+2n+2)/(n-k+1)$$

$$\times(\gamma+n-k)^{-1}\,G\begin{pmatrix}\gamma-a-k,\ \gamma-\beta-k,\\ n+\gamma-k+1,\ n-k+2,\end{pmatrix}\infty$$

$$=\Gamma(a+n+1)\,\Gamma(\beta+n+1)/\{\Gamma(n+\gamma-k+1)\,\Gamma(a+\beta-\gamma+2n+2)\}$$

$$\times(a-\gamma+k+1)_{n-k+1}\,(\beta-\gamma+k+1)_{n-k+1}/(n-k+1)!$$

$$\times G\begin{pmatrix}\gamma-a-k,\ \gamma-\beta-k,\\ n+\gamma-k+1,\ n-k+2,\end{pmatrix}\infty$$

$$=\ G\begin{pmatrix}\gamma-a-k,\ \gamma-\beta-k,\\ n+\gamma-k+1,\ n-k+2,\end{pmatrix}\infty$$

$$\times\Gamma(a+n+1)\,\Gamma(\beta+n+1)\,\Gamma(a-\gamma+n+2)\,\Gamma(\beta-\gamma+n+2),$$

divided by

$$\left[\begin{matrix}\Gamma(n-k+2)\,\Gamma(n+\gamma-k+1)\,\Gamma(a+\beta-\gamma+2n+2)\\ \times\Gamma(a-\gamma+k+1)\,\Gamma(\beta-\gamma+k+1)\end{matrix}\right].$$

Now, since k does not tend to infinity with n, the series G is near to unity.

It is therefore sufficient to find the value to which

$$\left.\begin{matrix}\Gamma(a+n+1)\,\Gamma(\beta+n+1)\,\Gamma(a-\gamma+n+2)\,\Gamma(\beta-\gamma+n+2)\\ \text{divided by}\\ \Gamma(n-k+2)\,\Gamma(n+\gamma-k+1)\,\Gamma(a+\beta-\gamma+2n+2)\end{matrix}\right\}\quad\text{(XVIII)}$$

tends as n increases.

Now omitting the factor $(2\pi)^{\frac{1}{2}}$, which is not here material, we may replace $\Gamma(x)$, when x is large, by $e^{-x}x^{x-\frac{1}{2}}$; and therefore $\Gamma(a+n)$, when n is large, by

$$e^{-a-n}\,(a+n)^{a+n-\frac{1}{2}},$$

i.e. $$e^{-a-n}n^{n+a-\frac{1}{2}}\left(1+\frac{a}{n}\right)^{n+a-\frac{1}{2}}.$$

Now $\left(1+\frac{a}{n}\right)^{n+a-\frac{1}{2}}$ tends to e^a as n tends to infinity.

Consequently we can replace $\Gamma(a+n)$ by $e^{-n}n^{n+a-\frac{1}{2}}$, and similarly $\Gamma(a+2n)$ by $e^{-2n}(2n)^{2n+a-\frac{1}{2}}$. Hence the expression (XVIII) can be replaced by

$$n^{2k+\alpha+\beta-2\gamma+\frac{1}{2}} \div 2^{2n+\alpha+\beta-\gamma+\frac{3}{2}},$$

which is small since k does not tend to infinity with n.

It should be observed that the $n-k+1$ terms of the series G in (XV), which go to make up a portion of the coefficient of $x^{1-\gamma+k}$ in

$$x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x),$$

are contributed by the first $n+1$ terms of the series

$$F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x).$$

However large k may be, we can take n so large that the coefficient of $x^{1-\gamma+k}$ is obtained to any required degree of accuracy.

This completes the examination of the first outstanding point noted in § 12, and shows that we may replace the $n-k+1$ terms of the series G in (XV) by the sum of that series extended to infinity.

15. It remains to examine the effect on the result of neglecting the terms after the $(n+1)$-th in the series (XII).

The $(p+1)$-th term of that series is

$$(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}.$$

Now $$(-1)^r/(1-\gamma-r)_{p+1} = 1/\{\gamma_r\,(1-\gamma)_{p+1-r}\}.$$

Hence, r being less than n, p greater than n, and γ a positive proper fraction, it appears that $(-1)^r/(1-\gamma-r)_{p+1}$ is positive.

Consequently every term in the expression

$$\sum_{r=0}^{n}\left[(-1)^r Rx^r/\{r!\,(n-r)!\}\right],$$

where

$$R = \left[\sum_{p=n+1}^{\infty}(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}\right]$$

is such that the coefficient of each power of x is positive.

Moreover, since $|x|<1$, the numerical value of the whole expression cannot exceed the value obtained by putting $x=1$.

We have therefore to calculate the value of

$$\sum_{r=0}^{n}[(-1)^r R/\{r!\,(n-r)!\}],$$

where

$$R=\Big[\sum_{p=n+1}^{\infty}(\alpha-\gamma+1)_p(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p(1-\gamma-r)_{p+1}\}\Big]. \tag{XIX}$$

Now

$$\sum_{r=0}^{n}[(-1)^r/\{r!\,(n-r)!\}][1/(1-\gamma-r)_{p+1}]$$

$$=\frac{1}{n!}\sum_{r=0}^{n}{}_nC_r(-1)^r/(1-\gamma-r)_{p+1}$$

$$=\frac{1}{n!}\sum_{r=0}^{n}(-1)^r\,{}_nC_r(1-\gamma-n)_{n-r}(2-\gamma-r+p)_r/(1-\gamma-n)_{p+1+n}$$

$$=\sum_{r=0}^{n}{}_nC_r(1-\gamma-n)_{n-r}(\gamma-p-1)_r/\{n!\,(1-\gamma-n)_{p+1+n}\}$$

$$=(-n-p)_n/\{n!\,(1-\gamma-n)_{p+1+n}\},$$

by (VI).

Hence the expression (XIX) is

$$\sum_{p=n+1}^{\infty}(\alpha-\gamma+1)_p(\beta-\gamma+1)_p(-n-p)_n/\{(1-\gamma-n)_{p+1+n}(\alpha+\beta-\gamma+n+2)_p\}, \tag{XX}$$

divided by $n!$.

Now the whole expression to be evaluated is the product of (XX) by

$$-\alpha_{n+1}\beta_{n+1}/\{\alpha+\beta-\gamma+1)_{n+1}(\gamma-1)_{n+1}\}$$

$$\times(\gamma-1)_{n+1}/n!.$$

But the expression (XX) is

$$(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}(-2n-1)_n/\{(1-\gamma-n)_{2n+2}(\alpha+\beta-\gamma+n+2)_{n+1}\} \tag{XXI}$$

multiplied by

$$1+\sum_{q=1}^{\infty}\left[\frac{(\alpha-\gamma+n+2)_q\,(\beta-\gamma+n+2)_q\,(2n+2)_q}{\{(n+2)_q\,(3-\gamma+n)_q\,(\alpha+\beta-\gamma+2n+3)_q\}}\right]. \qquad \text{(XXII)}$$

This series satisfies the condition of convergency.

All its terms are positive. Moreover,

$$\alpha+\beta-\gamma+2n+3>2n+2,$$

since $$1>\gamma-\alpha-\beta;$$

therefore $$(\alpha+\beta-\gamma+2n+3)_q>(2n+2)_q.$$

Hence the series (XXII) is less than the series

$$1+\sum_{q=1}^{\infty}(\alpha-\gamma+n+2)_q\,(\beta-\gamma+n+2)_q/\{(n+2)_q\,(3-\gamma+n)_q\}.$$

This series, which is

$$G\left(\begin{matrix}\alpha-\gamma+n+2, & \beta-\gamma+n+2, \\ 3-\gamma+n, & n+2,\end{matrix}\ \infty\right),$$

satisfies the condition of convergency.

If we apply to it the equation (V), it can be replaced by

$$(n+1)(\gamma-\alpha-\beta)^{-1}\,G\left(\begin{matrix}1-\alpha, & 1-\beta, \\ 3-\gamma+n, & \gamma-\alpha-\beta+1,\end{matrix}\ \infty\right).$$

Hence the whole expression to be evaluated is

$$-\alpha_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}\,(\gamma-1)_{n+1}\}$$

$$\times(\gamma-1)_{n+1}/n!$$

$$\times(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}(-2n-1)_n/\{(1-\gamma-n)_{2n+2}(\alpha+\beta-\gamma+n+2)_{n+1}$$

$$\times(n+1)(\gamma-\alpha-\beta)^{-1}\,G\left(\begin{matrix}1-\alpha, & 1-\beta, \\ 3-\gamma+n, & \gamma-\alpha-\beta+1,\end{matrix}\ \infty\right).$$

The factor $$-\alpha_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}\,(\gamma-1)_{n+1}\},$$

when n tends to infinity, tends to

$$\frac{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)}{\Gamma(\alpha)\,\Gamma(\beta)}.$$

The G series tends to 1, and $(\gamma-\alpha-\beta)^{-1}$ is finite.

Hence it remains only to show that

$$(n+1)(\gamma-1)_{n+1}/n!$$
$$\times(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}(-2n-1)_n,$$

divided by $\{(\alpha+\beta-\gamma+n+2)_{n+1}(1-\gamma-n)_{2n+2}\}$,

tends to zero as n tends to infinity.

This reduces to

$$-(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}/\{(2-\gamma)_{n+1}(\alpha+\beta-\gamma)_{n+1}\}$$
$$\times\quad(\alpha+\beta-\gamma)_{n+1}/(\alpha+\beta-\gamma+n+2)_{n+1}$$
$$\times\quad(2n+1)!/(n!\,n!). \qquad \text{(XXIII)}$$

The factor on the first line tends to

$$-\Gamma(2-\gamma)\,\Gamma(\alpha+\beta-\gamma)/\{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\},$$

as n tends to infinity.

The product of the factors on the second and third lines of (XXIII) is

$$\left[\frac{\Gamma(\alpha+\beta-\gamma+n+1)\,\Gamma(\alpha+\beta-\gamma+n+2)\,\Gamma(2n+2)}{[\Gamma(\alpha+\beta-\gamma)\,\Gamma(\alpha+\beta-\gamma+2n+3)\{\Gamma(n+1)\}^2]}\right]. \qquad \text{(XXIV)}$$

And now, as in Art. 14, replacing $\Gamma(\alpha+n)$ by $e^{-n}n^{n+\alpha-\frac{1}{2}}$, and $\Gamma(\alpha+2n)$ by $e^{-2n}(2n)^{2n+\alpha-\frac{1}{2}}$, and putting for brevity

$$\gamma-\alpha-\beta=\delta,$$

the expression (XXIV) is replaced by

$$2^{\delta-1}n^{-\delta}/\Gamma(-\delta),$$

which tends to zero as n tends to infinity, since δ is positive.

From this it follows that the terms in the G series in (XII) after the $(n+1)$-th may be neglected.

This completes the demonstration of the equation (I).

A NOTE ON A THEOREM OF MR. HARDY'S

By K. ANANDA RAU.

(*Communicated by* G. H. HARDY.)

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1. In a paper entitled "An extension of a theorem on oscillating series", published in Vol. 12 of these *Proceedings*,* Mr. Hardy has given the following theorem :

If Σc_n *is summable* (λ, κ), *and*

$$c_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then Σc_n *is convergent.*

Assuming that κ is an integer (which is clearly permissible), Mr. Hardy's argument enables us to prove successively that the series is summable $(\lambda, \kappa-1)$, $(\lambda, \kappa-2)$, ..., $(\lambda, 1)$; but it fails to prove the last step of this induction, namely, that if c_n satisfies the condition of the theorem, and if Σc_n is summable $(\lambda, 1)$, then the series is convergent. It is the object of this note to supply this last step and thus to complete the proof of the theorem.

2. Using the notation of Mr. Hardy's paper, what we have to prove is this :

If

$$\int^{\infty} \frac{B(\omega)}{\omega^2}\, d\omega$$

is convergent, then

$$(1) \quad B(\omega) = o(\omega).\dagger$$

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* Pp. 174–180.

† It is hardly necessary to point out that this is not a general convergence theorem true for all functions $B(\omega)$. It holds only in virtue of the structure of the special functions considered.

Suppose that (1) is false. Then there is a positive constant H, such that at least one of the inequalities

$$B(\omega) > H\omega,$$

$$B(\omega) < -H\omega$$

is satisfied for values of ω exceeding all limit. We shall shew that either of these hypotheses implies a contradiction. Let us take, for example, the first one.

Suppose that $\lambda_n \leqslant \omega < \lambda_{n+1}$. Let $\xi > \lambda_n$, and let $\lambda_{n+r} \leqslant \xi < \lambda_{n+r+1}$. Then

$$\begin{aligned} |B(\xi)-B(\omega)| &= |b_{n+1}+b_{n+2}+\ldots+b_{n+r}| \\ &< K(\lambda_{n+1}-\lambda_n+\lambda_{n+2}-\lambda_{n+1}+\ldots+\lambda_{n+r}-\lambda_{n+r-1}) \\ &= K(\lambda_{n+r}-\lambda_n) \\ &\leqslant K(\xi-\lambda_n). \end{aligned}$$

Further,

$$\begin{aligned} B(\xi) &\geqslant B(\omega)-|B(\xi)-B(\omega)| \\ &> H\omega-K(\xi-\lambda_n) \\ &= K\left(\lambda_n+\frac{H\omega}{K}-\xi\right). \end{aligned}$$

Hence, for

$$\lambda_n \leqslant \xi \leqslant \lambda_n+M\omega,$$

where

$$M = H/K,$$

we have

$$\frac{B(\xi)}{\xi^2} > \frac{K(\lambda_n+M\omega-\xi)}{\omega^2(1+M)^2};$$

and so

$$\begin{aligned} \int_{\lambda_n}^{\lambda_n+M\omega} \frac{B(\xi)}{\xi^2}\,d\xi &> \frac{K^3}{\omega^2(H+K)^2}\int_{\lambda_n}^{\lambda_n+M\omega}(\lambda_n+M\omega-\xi)\,d\xi \\ &= \frac{KH^2}{2(H+K)^2}. \end{aligned}$$

That this should be true for values of ω exceeding all limit is incom-

patible with the convergence of

$$\int^{\infty} \frac{B(\xi)}{\xi^2}\, d\xi.$$

The theorem is therefore proved.*

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* There is also a small slip on p. 177 of Mr. Hardy's paper. The inequality

$$\frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < \frac{K}{\kappa+1}(\xi-\omega)\xi^{\kappa},$$

should be

$$\frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < K(\xi-\omega)\xi^{\kappa}.$$

The succeeding argument remains valid if the factor $\kappa+1$ is omitted in the three places where it occurs in a denominator.

ON INVARIANTS AND COVARIANTS OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

By E. B. STOUFFER.

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IT is known* that the most general transformations which convert the system of linear homogeneous differential equations

$$y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0 \quad (i = 1, 2, \ldots, n), \tag{1}$$

where p_{ik} and q_{ik} are functions of the independent variable x, into another of the same form, are given by the equations

$$y_\kappa = \sum_{\lambda=1}^{n} a_{\kappa\lambda}(x)\,\eta_\lambda \quad (\kappa = 1, 2, \ldots, n), \tag{2}$$

$$\xi = \xi(x), \tag{3}$$

where $a_{\kappa\lambda}$ and ξ are arbitrary functions of x, for which the determinant

$$|a_{\kappa\lambda}| \quad (\kappa, \lambda = 1, 2, \ldots, n)$$

does not vanish identically. A function of the coefficients of (1) and their derivatives, and of the dependent variables and their derivatives, which has the same value for (1) as for any system derived from (1) by the transformations (2), is called a *semi-covariant*; if the function keeps the same value when (1) is transformed by (3) also, it is called a *covariant*. A semi-covariant or a covariant is called a *seminvariant* or an *invariant*, respectively, if the dependent variables and their derivatives do not occur in the function.

In a preceding paper† the writer has obtained a complete system of seminvariants for the system (1). It is the object of the present paper to complete the problem by the calculation of complete systems of invariants,

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906, Chap. I.

† "On Seminvariants of Linear Homogeneous Differential Equations", 1916, *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. 217.

semi-covariants, and covariants. The methods used largely avoid the solution of the complicated systems of partial differential equations which arise by the Lie theory.

By means of the system (1) and its invariants and covariants we can study the projective differential properties of any spread (except for special cases) generated by ∞^1 flats in space of any number of dimensions. It is thus unnecessary to employ systems of higher order. This geometry will be discussed in papers to follow.

1. *The Invariants.*

As invariants are also seminvariants, the calculation of invariants merely requires the discovery of functions of the seminvariants which keep the same value after the system (1) is transformed in accordance with equation (3).

In the calculation of the seminvariants, it was found convenient to introduce auxiliary functions of the coefficients of (1) and of their derivatives, denoted by

$$u_{ik},\ v_{ik},\ w_{ik} \quad (i,\ k = 1,\ 2,\ \ldots,\ n),$$

and defined by the equations

$$\left.\begin{aligned} u_{ik} &= p'_{ik} - q_{ik} + \sum_{j=1}^{n} p_{ij}\,p_{jk} \\ v_{ik} &= u'_{ik} + \sum_{j=1}^{n} (p_{ij}\,u_{jk} - p_{jk}\,u_{ij}) \\ w_{ik} &= v'_{ik} + \sum_{j=1}^{n} (p_{ij}\,v_{ik} - p_{jk}\,v_{ij}) \end{aligned}\right\} \quad (i,\ k = 1,\ 2,\ \ldots,\ n). \tag{4}$$

If I is a seminvariant involving u_{ik}, r successive applications of the operator

$$D_u = \sum_{i=1}^{n} \frac{\partial}{\partial u_{ii}}$$

upon I are indicated by

$$I^{(r)} = D_u^r I = \frac{1}{r!} \sum_{i=1}^{n} \left(\frac{\partial}{\partial u_{ii}}\right)^r I.$$

Similarly, successive applications of the operators

$$D_{uv} = \sum_{i=1}^{n} \sum_{k=1}^{n} v_{ik} \frac{\partial}{\partial u_{ik}},$$

$$D_{vw} = \sum_{i=1}^{n} \sum_{k=1}^{n} w_{ik} \frac{\partial}{\partial v_{ik}}$$

are indicated by

$$I^{(rs)} = D_{uv}^{s} I^{(r)} = \frac{1}{s!} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(v_{ik} \frac{\partial}{\partial u_{ik}} \right)^{s} I^{(r)},$$

$$I^{(rst)} = D_{vw}^{t} I^{(rs)} = \frac{1}{t!} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(w_{ik} \frac{\partial}{\partial v_{ik}} \right)^{t} I^{(rs)},$$

respectively.

For the purpose of the calculation of the invariants it is convenient to replace the complete system of seminvariants* consisting of

$$(5) \quad \begin{cases} I^{(rs)} (r = 0, 1, \ldots, n-1;\ s = 0, 1, 2, \ldots;\ r+s \leqslant n),\ I_1^{(rs)} (s > 2), \\ I^{(rs1)} (s > 0),\quad D_w^t J \ (t = 0, 1, \ldots, n-2), \end{cases}$$

and certain derivatives, by the system consisting of

$$(6) \quad \begin{cases} I^{(r00)} = I^{(r)} \ (r = 0, 1, \ldots, n-1), \\ I^{(rs0)} = I^{(rs)} \ (r = 0, 1, \ldots, n-1;\ s = 1, 2, \ldots;\ r+s \leqslant n), \\ I^{(rst)} \ (r = 0, 1, \ldots, n-1;\ s = 1, 2, \ldots;\ r+s \leqslant n;\ t = 1, 2, 3;\ t \leqslant s), \end{cases}$$

and certain derivatives.

These seminvariants are proved to be independent by methods very similar to those by which the seminvariants (5) were proved to be independent. In fact, it is only necessary to find a non-vanishing determinant of maximum order in their functional matrix. First put

$$u_{ik} = 0 \ (i \neq k).$$

It is then easy to see that there exists a determinant of maximum order which has for one factor the cube of the non-vanishing Jacobian of the elementary symmetric functions of $u_{11}, u_{22}, \ldots, u_{nn}$. The other factor F is a determinant in which the coefficient of the highest power of u_{nn} is different from zero. In fact, this coefficient is the product of the determinant which corresponds to F, in the case of $n-1$ variables in equations (1), and the functional determinant of $I^{(rst)}$ $(r+s = n,\ t = 0, 1, 2, 3)$ with respect to all variables v_{ik}, w_{ik} for which $i = n$ or $k = n$, except $v_{n,n-1}$, v_{nn}, w_{nn}. If we arrange the rows and columns in proper order in this latter determinant, and put w_{in} $(i = 1, 2, \ldots, n-2)$ and all v_{ik} except $v_{i,i-1}$ $(i = 2, 3, \ldots, n)$ equal to zero, the determinant has all elements of the principle diagonal different from zero and all elements above the

* Stouffer, *loc. cit.*, p. 226.

principle diagonal equal to zero. Thus this factor does not vanish, and the whole functional determinant is different from zero, provided it is non-vanishing in the case of $n-1$ variables in (1). But it is easily seen to be different from zero in the case of $n = 3$. Hence by induction the seminvariants are proved to be independent.

The system (6) contains $2n^2+1$ seminvariants. But the system (5) also contains $2n^2+1$ independent seminvariants, and these with the successive derivatives of the n^2 seminvariants of (5) which contain w_{ik} form a complete system. Therefore the system (6) and the successive derivatives of any n^2 of the seminvariants $I^{(rst)}$ $(t = 1, 2, 3)$ form a complete system. We shall use the successive derivatives of the n^2 seminvariants for which $t = 1, 2$.

In order to calculate the invariants we shall need to find the effect of the transformation (3) upon the seminvariants. By this transformation equations (1) are converted into

$$\frac{d^2y_i}{d\bar{\xi}^2} + \sum_{k=1}^{n} \left(2\bar{p}_{ik}\frac{dy_k}{d\bar{\xi}} + \bar{q}_{ik}y_k\right) = 0 \quad (i = 1, 2, \ldots, n),$$

where

$$(7) \qquad \begin{cases} \bar{p}_{ii} = \dfrac{1}{\bar{\xi}'}(p_{ii}+\tfrac{1}{2}\eta), \\ \bar{p}_{ik} = \dfrac{1}{\bar{\xi}'}p_{ik} \quad (i \neq k), \\ \bar{q}_{ik} = \dfrac{1}{(\bar{\xi}')^2}q_{ik}, \end{cases}$$

with

$$(8) \qquad \eta = \frac{\bar{\xi}''}{\bar{\xi}'}.$$

If the transformation (3) is made infinitesimal by putting

$$(9) \qquad \bar{\xi}(x) = x+\phi(x)\,\delta t,$$

where $\phi(x)$ is an arbitrary function of x and δt an infinitesimal, we have, on neglecting higher powers of δt than the first,

$$(10) \qquad \begin{cases} \delta p_{ii} = \bar{p}_{ii}-p_{ii} = (-\phi' p_{ii}+\tfrac{1}{2}\phi'')\,\delta t, \\ \delta p_{ik} = \bar{p}_{ik}-p_{ik} = -\phi' p_{ik}\delta t \quad (i \neq k), \\ \delta q_{ik} = \bar{q}_{ik}-q_{ik} = -2\phi' q_{ik}\delta t\,; \end{cases}$$

whence, by substitution in (4),

$$(11)\qquad \left\{\begin{aligned}
\delta u_{ii} &= (-2\phi' u_{ii} + \tfrac{1}{2}\phi^{(3)})\delta t,\\
\delta u_{ik} &= -2\phi' u_{ik}\delta t \quad (i \neq k),\\
\delta v_{ii} &= (-3\phi' v_{ii} - 2\phi'' u_{ii} + \tfrac{1}{2}\phi^{(4)})\delta t,\\
\delta v_{ik} &= (-3\phi' v_{ik} - 2\phi'' u_{ik})\delta t \quad (i \neq k),\\
\delta w_{ii} &= (-4\phi' w_{ii} - 5\phi'' v_{ii} - 2\phi^{(3)} u_{ii} + \tfrac{1}{2}\phi^{(5)})\delta t,\\
\delta w_{ik} &= (-4\phi' w_{ik} - 5\phi'' v_{ik} - 2\phi^{(3)} u_{ik})\delta t \quad (i \neq k).
\end{aligned}\right.$$

From these values for δu_{ik} and the definition of $I^{(r)}$, it follows that

$$\delta I^{(r)} = \sum_{u_{ik}} \frac{\partial I^{(r)}}{\partial u_{ik}} \delta u_{ik} = \sum_{u_{ik}} \frac{\partial I^{(r)}}{\partial u_{ik}} (-2\phi' u_{ik}\delta t) + \sum_{u_{ii}} \frac{\partial I^{(r)}}{\partial u_{ii}} (\tfrac{1}{2}\phi^{(3)}\delta t),$$

or

$$(12)\qquad \delta I^{(r)} = -2(n-r)\,\phi' I^{(r)}\delta t + \tfrac{1}{2}(r+1)\,\phi^{(3)} I^{(r+1)}\delta t.$$

If f is an invariant, the system of partial differential equations* which it must satisfy is obtained by equating to zero the symbols of the infinitesimal transformations of f. If f depends only on $I^{(0)}, I^{(1)}, \ldots, I^{(n-1)}$, equations (12) show that it must satisfy

$$(13)\qquad \left\{\begin{aligned}
&\sum_{j=0}^{n} (j+1)\, I^{(j+1)} \frac{\partial f}{\partial I^{(j)}} = 0,\\
&\sum_{j=0}^{n} (n-j)\, I^{(j)} \frac{\partial f}{\partial I^{(j)}} = 0.
\end{aligned}\right.$$

We proceed to find all these invariants. The second equation of (13) merely requires that absolute invariants be isobaric of weight zero, or relative invariants be isobaric. Since there are n independent variables and two independent equations in the system (13), there are $n-2$ absolute invariants or $n-1$ relative invariants here. The method of undetermined coefficients shows that two isobaric functions which satisfy the first equation of (13) are

$$\Theta_2 = (I^{(n-1)})^2 - \frac{2n}{n-1} I^{(n-2)},$$

$$\Theta_3 = (I^{(n-1)})^3 - 3I^{(n-1)}\Theta_2 - \frac{6n^2}{(n-1)(n-2)} I^{(n-3)}.$$

* Lie, *Mathematische Annalen*, 1884, Vol. 24, p. 537.

It is now easily shown by induction that the first equation of (13) is satisfied by the isobaric functions

$$\Theta_\alpha = (I^{(n-1)})^\alpha - \alpha! \sum_{i=2}^{\alpha-1} \frac{1}{(\alpha-i)!\, i!} (I^{(n-1)})^{\alpha-i} \Theta_i$$

$$- \frac{\alpha!\, n^{\alpha-1}}{(n-1)(n-2)\dots(n-\alpha+1)} I^{(n-\alpha)} \quad (\alpha = 2, 3, \dots, n).$$

These $n-1$ expressions are relative invariants whose respective weights are twice the subscript of Θ. They are independent since each contains an independent seminvariant not contained in those of lower weight. Since there are only $n-1$ such relative invariants, we have them all.

We next proceed to find the invariants involving $I^{(rs)}$ $(s > 0)$. If $\Theta_{\alpha+\beta}$ denotes such a relative invariant, homogeneous of degree α in u_{ik} and of degree β in v_{ik}, that is, isobaric of weight 2α in μ_{ik} and isobaric of weight 3β in v_{ik}, the transformation (3) converts it into $\bar{\Theta}_{\alpha+\beta}$, where

$$\bar{\Theta}_{\alpha+\beta} = \frac{1}{(\xi')^{2\alpha+3\beta}} \Theta_{\alpha+\beta},$$

and the infinitesimal transformation (9) gives to $\Theta_{\alpha+\beta}$ the increment

$$\delta\Theta_{\alpha+\beta} = -(2\alpha+3\beta)\, \phi' \Theta_{\alpha+\beta} \delta t.$$

It follows that

$$\delta(D_{uv}\Theta_{\alpha+\beta}) = \sum_{i=1}^{n} \sum_{k=1}^{n} v_{ik} \frac{\partial}{\partial u_{ik}} \left[\delta\Theta_{\alpha+\beta} - \delta u_{ik} \frac{\partial}{\partial u_{ik}} \Theta_{\alpha+\beta} \right] + \sum_{i=1}^{n} \sum_{k=1}^{n} \delta v_{ik} \frac{\partial}{\partial u_{ik}} \Theta_{\alpha+\beta}$$

$$= \{ -(2\alpha+3\beta-2)\, \phi' D_{uv}\Theta_{\alpha+\beta}$$

$$+ (-3\phi' D_{uv}\Theta_{\alpha+\beta} - 2\alpha\phi'' \Theta_{\alpha+\beta} + \tfrac{1}{2}\phi^{(4)} D_u \Theta_{\alpha+\beta}) \}\, \delta t,$$

$$(14) \quad \frac{\delta(D_{uv}\Theta_{\alpha+\beta})}{\delta t} = -(2\alpha+3\beta+1)\, \phi' D_{uv}\Theta_{\alpha+\beta} - 2\alpha\phi'' \Theta_{\alpha+\beta} + \tfrac{1}{2}\phi^{(4)} D_u \Theta_{\alpha+\beta}.$$

Now $$D_u \Theta_2 = 0,$$

whence by induction $D_u\Theta_\alpha = 0$ $(\alpha = 2, 3, \dots, n)$.

Therefore, if we put $\beta = 0$ in (14), we have

$$\frac{\delta(D_{uv}\Theta_\alpha)}{\delta t} = -(2\alpha+1)\, \phi' D_{uv}\Theta_\alpha - 2\alpha\phi'' \Theta_\alpha.$$

Consequently, the isobaric expressions

$$\Theta_{\alpha,1} = 2\Theta_2 D_{uv}\Theta_\alpha - \alpha\Theta_\alpha D_{uv}\Theta_2 \quad (\alpha = 3, 4, \ldots, n), \tag{15}$$

are relative invariants of weight $2\alpha+2$ in u_{ik} and 3 in v_{ik}.

A more general form of (15) is obtained if we notice that, according to (14),

$$\Theta_{\alpha+\beta,1} = 2\Theta_2 D_{uv}\Theta_{\alpha+\beta} - \alpha\Theta_{\alpha+\beta} D_{uv}\Theta_2 \tag{16}$$

is a relative invariant provided that

$$D_u\Theta_{\alpha+\beta} = 0.$$

Since

$$D_u(D^r_{uv}\Theta_\alpha) = D^r_{uv}(D_u\Theta_\alpha) = 0, \tag{17}$$

we have

$$D_u\Theta_{\alpha,1} = 0.$$

Therefore

$$\Theta_{\alpha,2} = 2\Theta_2 D_{uv}\Theta_{\alpha,1} - (\alpha+1)\,\Theta_{\alpha,1} D_{uv}\Theta_2 \quad (\alpha = 3, 4, \ldots, n)$$

are invariants. Again, by (17),

$$D_u\Theta_{\alpha,2} = 0,$$

whence

$$\Theta_{\alpha,3} = 2\Theta_2 D_{uv}\Theta_{\alpha,2} - (\alpha+2)\,\Theta_{\alpha,2} D_{uv}\Theta_2 \quad (\alpha = 3, 4, \ldots, n)$$

are invariants. In fact a continuation of this reasoning gives the $\frac{n^2+n-6}{2}$ relative invariants

$$\Theta_{\alpha,\beta} = 2\Theta_2 D_{uv}\Theta_{\alpha,\beta-1} - (\alpha+\beta-1)\,\Theta_{\alpha,\beta-1} D_{uv}\Theta_2$$
$$(\alpha = 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, \alpha),$$

whose respective weights are $2\alpha+2\beta$ in u_{ik} and 3β in v_{ik}.

The invariants Θ_α, $\Theta_{\alpha,\beta}$ are $\frac{n^2+3n-8}{2}$ in number. Moreover they are independent, since each contains a seminvariant not in those which precede when they are arranged in order of ascending values of α and β. The system of partial differential equations which these invariants must satisfy contains 4 equations and $\frac{n^2+3n}{2}$ variables $I^{(rs)}$. There are, therefore,

$$\frac{n^2+3n}{2} - 4 + 1 = \frac{n^2+3n-6}{2}$$

independent relative invariants here. We have all but one. It is easily found by the method of undetermined coefficients that

$$\Im_{0,2} = 4\Theta_2(I^{(n-1,1)})^2 - \frac{8n}{n-1} I^{(n-2,2)}\Theta_2 - (\Theta_2')^2$$

is a relative invariant of weight 10. That it is independent of the other invariants is seen by arranging them in the order Θ_α, $\Im_{0,2}$, $\Theta_{\alpha,\beta}$, and in order of ascending values of α and β.

We continue the process of finding the invariants by seeking those which involve $I^{(rst)}$ $(t = 1, 2, 3)$. It may be verified easily by direct substitution that the expressions

$$\Im_{2\alpha+1} = 4\alpha\Theta_\alpha''\Theta_\alpha - (4\alpha+1)(\Theta_\alpha')^2 + \frac{16\alpha^2}{n} I^{(n-1)}\Theta_\alpha^2,$$

called the quadriderivatives* of Θ_α, are relative invariants. The quadriderivatives of Θ_2 and Θ_3

$$\Im_5 = 8\Theta_2''\Theta_2 - 9(\Theta_2')^2 + \frac{64}{n} I^{(n-1)}\Theta_2^2,$$

$$\Im_7 = 12\Theta_3''\Theta_3 - 13(\Theta_3')^2 + \frac{144}{n} I^{(n-1)}\Theta_3^2,$$

contain $I^{(n-1,1,1)}$, $I^{(n-2,1,1)}$, and $I^{(n-3,1,1)}$, but no other seminvariants $I^{(rst)}$ $(t > 0)$. For Θ_2 and Θ_3 contain only $I^{(n-1)}$, $I^{(n-2)}$, and I^{n-3}, whose second derivatives expressed in terms of seminvariants are respectively

$$I^{(n-1,1,1)}, \quad 2I^{(n-2,2)} + I^{(n-2,1,1)}, \quad 2I^{(n-3,2)} + I^{(n-3,1,1)}.$$

It is not difficult to verify the extended form of equation (14),

$$\frac{\delta(D_{vw}\Theta_{\alpha,\beta})}{\delta t} = -\phi'(2\alpha+5\beta+1) D_{vw}\Theta_{\alpha,\beta} - 5\beta\phi''\Theta_{\alpha,\beta} - 2\phi^{(3)} D_{vu}\Theta_{\alpha,\beta} + \tfrac{1}{2}\phi^{(5)} D_v\Theta_{\alpha,\beta}. \tag{18}$$

Moreover, by induction, it is easily shown that

$$D_{vu}\Theta_{\alpha,\beta} = 0, \quad D_v\Theta_{\alpha,\beta} = 0.$$

Consequently (18) becomes

$$\frac{\delta(D_{vw}\Theta_{\alpha,\beta})}{\delta t} = -\phi'(2\alpha+5\beta+1) D_{vw}\Theta_{\alpha,\beta} - 5\beta\phi''\Theta_{\alpha,\beta}. \tag{19}$$

* Wilczynski, *loc. cit.*, p. 112.

Therefore, the expressions

$$\Theta_{\alpha,\beta,1} = \Theta_{3,1} D_{vw} \Theta_{\alpha,\beta} - \beta \Theta_{\alpha,\beta} D_{vw} \Theta_{3,1} \tag{20}$$

$$(\alpha = 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, a),$$

are relative invariants of weights $2\alpha+5\beta+12$. Of course $\Theta_{3,1,1}$ vanishes identically. Moreover, since

$$D_{vu}(D^{l}_{vw}\Theta_{\alpha,\beta}) = 0, \qquad D_{v}(D^{l}_{vw}\Theta_{\alpha,\beta}) = 0,$$

we have $$\frac{\delta(D_{vw}\Theta_{\alpha,\beta,1})}{\delta t} = -\phi'(2\alpha+5\beta+13) D_{vw}\Theta_{\alpha,\beta,1} - 5\beta\phi''\Theta_{\alpha,\beta,1},$$

whence it follows that

$$\Theta_{\alpha,\beta,2} = \Theta_{3,1} D_{vw} \Theta_{\alpha,\beta,1} - \beta \Theta_{\alpha,\beta,1} D_{vw} \Theta_{3,1}$$

is an invariant of weight $2\alpha+5\beta+24$. A continuation of this process shows that

$$\Theta_{\alpha,\beta,\gamma} = \Theta_{3,1} D_{vw} \Theta_{\alpha,\beta,\gamma-1} - \beta \Theta_{\alpha,\beta,\gamma-1} D_{vw} \Theta_{3,1}$$

$$(\alpha = 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, a;\ \gamma = 1, 2, \ldots, \beta),$$

are invariants of weight $2\alpha+5\beta+12\gamma$, respectively. If only the seminvariants (6) are to be included, γ must be limited, to the values 1, 2, 3.

Again, $$\frac{\delta(D_{vw}\Im_{0,2})}{\delta t} = -11\phi' D_{vw}\Im_{0,2} - 10\phi''\Im_{0,2},$$

which shows that

$$\Im_{0,3} = 2\Theta_2 D_{vw}\Im_{0,2} - 5\Theta_2'\Im_{0,2},$$

$$\Im_{0,4} = 4\Theta_2(2\Theta_2 D^2_{vw}\Im_{0,2} - 5\Theta_2' D_{vw}\Im_{0,2}) + 25(\Theta_2')^2\Im_{0,2}$$

are relative invariants.

If the invariants

$$\Theta_\alpha,\ \Im_{0,2},\ \Theta_{\alpha,\beta},\ \Im_5,\ \Im_7,\ \Im_{0,3},\ \Im_{0,4},\ \Theta_{\alpha,\beta,\gamma} \tag{21}$$

$$(\alpha = 2, 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, a;\ \gamma = 1, 2, 3;\ \gamma \leqslant \beta)$$

are arranged in this order and in order of ascending values of α, β, γ, they are seen to be independent since each contains a seminvariant $I^{(rst)}$ not in those which precede.

In each invariant the weight is twice the first subscript plus five times the second, if it occurs, plus twelve times the third, if it occurs.

The system of partial differential equations which the invariants

depending on the seminvariants (6) must satisfy contains five independent equations and $2n^2+1$ variables, and has, therefore, $2n^2-4$ solutions. Thus there are $2n^2-3$ such relative invariants. The set (21) contains the required number of independent relative invariants.

The complete system of seminvariants involves, in addition to the seminvariants (6), the successive derivatives of n^2 of the seminvariants $I^{(rst)}$, say those for which $t = 1, 2$. The first derivative thus introduces n^2 new seminvariants, and the system of partial differential equations for the invariants has one more equation than before. Thus there are n^2-1 new relative invariants.

To obtain these let us notice that, from two relative invariants Θ_μ, Θ_ν of weights μ and ν, respectively, we can obtain a relative invariant

$$\mu\Theta_\mu\Theta'_\nu-\nu\Theta_\nu\Theta'_\mu,$$

the so called Jacobian* of Θ_μ and Θ_ν. Thus the combination by the Jacobian process of some invariant, say Θ_2, with the n^2-1 invariants ϑ_5, ϑ_7, $\vartheta_{0,3}$, $\vartheta_{0,4}$, $\Theta_{\alpha,\beta,1}$, $\Theta_{\alpha,\beta,2}$ gives n^2-1 new relative invariants involving first derivatives of $I^{(rs1)}$ and $I^{(rs2)}$. Moreover they must be independent of each other and of the invariants (21). For, inasmuch as these new invariants are merely combinations of the invariants (21) and their derivatives, a relation involving them would imply a relation between the invariants (21).

The n^2-1 invariants involving the next higher derivatives of $I^{(rst)}$ $(t = 1, 2)$ can be obtained by combining Θ_2 by the Jacobian process with the n^2-1 just obtained. A continuation of this process gives the relative invariants involving derivatives to as high an order as desired.

We thus have proved that:

The invariants (21) *and those obtained from them by the Jacobian process form a system of invariants, complete in the sense that all invariants can be expressed in terms of them.*

2. *Semi-Covariants.*

It is not necessary to seek semi-covariants which involve higher derivatives of y_i than the first, for such higher derivatives may be removed by means of equation (1). Since seminvariants are also semi-covariants, the systems of partial differential equations which semi-covariants must satisfy are the same as the systems which the

* Wilczynski, *loc. cit.*, p. 112.

seminvariants must satisfy, with the addition of the terms which arise from the infinitesimal transformation of y_i and y_i'. Thus to the system* which determines the seminvariants involving p_{ik}, p_{ik}', p_{ik}'', q_{ik}, q_{ik}' is added the $2n$ variables y_i and y_i' but no new equations. However, the one relation between those equations ceases to hold, so that there are only $2n-1$ more solutions, that is, there are $2n-1$ absolute semi-covariants, or $2n$ relative semi-covariants, which are not seminvariants. Moreover, the systems which determine the semi-covariants involving higher derivatives of p_{ik} and q_{ik} show that there are no more independent semi-covariants. We proceed now to find $2n$ independent semi-covariants.

The transformation (2) may be made infinitesimal by putting

$$a_{ii}(x) = 1+\phi_{ii}(x)\,\delta t, \quad a_{ik}(x) = \phi_{ik}(x)\,\delta t \quad (i \neq k\,;\ i, k = 1, 2, \ldots, n),$$

where δt is an infinitesimal and the ϕ_{ik}'s are arbitrary functions of x. The infinitesimal transformations of y_i, p_{ik}, u_{ik}, v_{ik} are then found by direct substitution to be

$$\left.\begin{aligned}
\frac{\delta y_i}{\delta t} &= -\sum_{j=1}^{n} \phi_{ij} y_j \\
\frac{\delta p_{ik}}{\delta t} &= \sum_{j=1}^{n} (\phi_{jk} p_{ij} - \phi_{ij} p_{jk}) + \phi_{ik}' \\
\frac{\delta u_{ik}}{\delta t} &= \sum_{j=1}^{n} (\phi_{jk} u_{ij} - \phi_{ij} u_{jk}) \\
\frac{\delta v_{ik}}{\delta t} &= \sum_{j=1}^{n} (\phi_{jk} v_{ij} - \phi_{ij} v_{jk})
\end{aligned}\right\} \quad (i, k = 1, 2, \ldots, n).$$

The expressions u_{ik} and v_{ik} are therefore cogredient.

The quantities $$r_i^{(1)} = \sum_{j=1}^{n} u_{ij} y_j \quad (i = 1, 2, \ldots, n)$$

are cogredient with y_i. For

$$\begin{aligned}
\frac{\delta r_i^{(1)}}{\delta t} &= \sum_{j=1}^{n} \left[u_{ij} \sum_{\lambda=1}^{n} (-\phi_{j\lambda} y_\lambda) + y_j \sum_{\lambda=1}^{n} (\phi_{\lambda j} u_{i\lambda} - \phi_{i\lambda} u_{\lambda j}) \right] \\
&= -\sum_{j=1}^{n} \phi_{ij} \sum_{\lambda=1}^{n} u_{j\lambda} y_\lambda \\
&= -\sum_{j=1}^{n} \phi_{ij} r_j^{(1)}.
\end{aligned}$$

* Stouffer, *loc. cit.*, pp. 221, 222.

It follows that each of the $n-2$ sets of quantities

$$\left.\begin{aligned} r_i^{(2)} &= \sum_{j=1}^{n} u_{ij} r_j^{(1)} \\ r_i^{(3)} &= \sum_{j=1}^{n} u_{ij} r_j^{(2)} \\ &\cdots \quad \cdots \quad \cdots \\ r_i^{(n-1)} &= \sum_{j=1}^{n} u_{ij} r_j^{(n-2)} \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

are also cogredient with y_i. Therefore the determinant

$$R = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ r_1^{(1)} & r_2^{(1)} & \cdots & r_n^{(1)} \\ r_1^{(2)} & r_2^{(2)} & \cdots & r_n^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{(n-1)} & r_2^{(n-2)} & \cdots & r_n^{(n-1)} \end{vmatrix}$$

is a semi-covariant.

Again, since u_{ik} and v_{ik} are cogredient, the quantities

$$s_i = \sum_{j=1}^{n} v_{ij} y_j \quad (i = 1, 2, \ldots, n)$$

are cogredient with y_i. Therefore the $n-1$ determinants

$$S_i = \sum_{j=1}^{n} s_j \frac{\partial R}{\partial r_j^{(i)}} \quad (i = 1, 2, \ldots, n-1)$$

are relative semi-covariants.

In order to prove that R and S_i are independent of each other and of the seminvariants, we need only show that the functional determinant

$$\frac{\partial(R, S_1, S_2, \ldots, S_{n-1})}{\partial(y_1, y_2, y_3, \ldots, y_n)} \tag{22}$$

does not vanish identically. If we put $u_{ik} = 0$ $(i \neq k)$, the highest power of u_{nn} in R and in all S_i except S_{n-1} occurs in each semi-covariant in the form $u_{nn}^{n-1} y_n$, multiplied by the corresponding semi-covariant for the case of $n-1$ variables in equations (1). Moreover, u_{nn}^{n-1} does not occur in S_{n-1}. Hence in the functional determinant above the highest power of u_{nn} occurs in the form $(u_{nn}^{n-1} y_n)^{n-1}$, multiplied by the partial derivative of S_{n-1} with respect to y_n, and by the functional determinant

corresponding to (22), with $u_{ik} = 0$ $(i \neq k)$, in the case of $n-1$ variables in (1). If the latter determinant is different from zero, (22) cannot vanish identically. But (22), with $u_{ik} = 0$ $(i \neq k)$, does not vanish identically for $n = 2$, as is easily verified, and it follows by induction that (22) does not vanish identically.

Again the quantities $\quad t_i = y_i' + \sum_{j=1}^{n} p_{ij} y_j \quad (i = 1, 2, \ldots, n)$

are cogredient with y_i. For

$$\frac{\delta t_i}{\delta t} = -\sum_{j=1}^{n} (\phi_{ij}' y_j + \phi_{ij} y_j')$$

$$+ \sum_{j=1}^{n} \left[p_{ij} \sum_{\lambda=1}^{n} (-\phi_{j\lambda} y_\lambda) + y_j \sum_{\lambda=1}^{n} (\phi_{\lambda j} p_{i\lambda} - \phi_{i\lambda} p_{\lambda j}) + y_j \phi_{ij}' \right]$$

$$= -\sum_{j=1}^{n} \phi_{ij} \left[y_j' + \sum_{\lambda=1}^{n} p_{j\lambda} y_\lambda \right]$$

$$= -\sum_{j=1}^{n} \phi_{ij} t_j.$$

Hence the n determinants

$$T_y = \sum_{j=1}^{n} t_j \frac{\partial R}{\partial y_j},$$

$$T_i = \sum_{j=1}^{n} t_j \frac{\partial R}{\partial r_j^{(i)}} \quad (i = 1, 2, \ldots, n-1),$$

are semi-covariants.

To prove that T_y and T_i are independent of each other and of R, S_i and the seminvariants it is only necessary to show that the determinant

$$\frac{\partial (T_y, T_1, T_2, \ldots, T_{n-1})}{\partial (y_1', y_2', y_3', \ldots, y_n')}$$

does not vanish identically. If we put $u_{ik} = 0$ $(i \neq k)$, this determinant becomes the product of $(y_1 y_2 y_3 \ldots y_n)^{n-1}$ and the reciprocal of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ u_{11} & u_{22} & u_{33} & \ldots & u_{nn} \\ u_{11}^2 & u_{22}^2 & u_{33}^2 & \ldots & u_{nn}^2 \\ \ldots & \ldots & \ldots & & \ldots \\ u_{11}^{n-1} & u_{22}^{n-1} & u_{33}^{n-1} & \ldots & u_{nn}^{n-1} \end{vmatrix}.$$

It does not vanish identically, therefore, and the semi-covariants thus far found are independent.

We thus have just the required $2n$ independent relative semi-covariants and need seek no further.

3. *The Covariants.*

The covariants are those functions of the semi-covariants (including seminvariants) which are unchanged in value after (1) is transformed by (3). Since there are just $2n-1$ absolute semi-covariants, there can be only $2n-1$ more absolute covariants than absolute invariants, and $2n$ more relative covariants than relative invariants.

If the transformation (3) is assumed to be in the form (9), equations (11) show that the infinitesimal transformations of $r_i^{(1)}$, $r_i^{(2)}$, ... are

$$\left.\begin{aligned} \frac{\delta r_i^{(1)}}{\delta t} &= -2\phi' r_i^{(1)} + \tfrac{1}{2}\phi^{(3)} y_i \\ \frac{\delta r_i^{(2)}}{\delta t} &= -4\phi' r_i^{(2)} + \phi^{(3)} r_i^{(1)} \\ \frac{\delta r_i^{(3)}}{\delta t} &= -6\phi' r_i^{(3)} + \tfrac{3}{2}\phi^{(3)} r_i^{(2)} \end{aligned}\right\} \quad (i = 1, 2, \ldots, n);$$

whence, by induction,

$$\frac{\delta r_i^{(l)}}{\delta t} = -2l\phi' r_i^{(l)} + \tfrac{1}{2} l \phi^{(3)} r_i^{(l-1)} \tag{23}$$

$$(i = 1, 2, \ldots, n;\ l = 1, 2, \ldots, n-1),$$

where we have put $r_i^{(0)} = y_i$. Also, from (11),

$$\frac{\delta s_i}{\delta t} = -3\phi' s_i - 2\phi' r_i^{(1)} + \tfrac{1}{2}\phi^{(4)} y_i \quad (i = 1, 2, \ldots, n). \tag{24}$$

Finally, from (10), and the fact that

$$\frac{\delta y_i'}{\delta t} = -\phi' y_i, \tag{25}$$

we have

$$\frac{\delta t_i}{\delta t} = -\phi' t_i + \tfrac{1}{2}\phi'' y_i \quad (i = 1, 2, \ldots, n). \tag{26}$$

The results given by equations (23), (24), (25), and (26) make it possible to calculate the infinitesimal transformations of the covariants

easily. Thus

$$\frac{\delta R}{\delta t} = \sum_{k=1}^{n} \sum_{l=1}^{n-1} (-2l\phi' r_k^{(l)} + \tfrac{1}{2} l \phi^{(3)} r_k^{(l-1)}) \frac{\partial R}{\partial r_k^{(l)}}$$

$$= -n(n-1)\,\phi' R,$$

and R is therefore a relative covariant. Again,

$$\frac{\delta S_i}{\delta t} = \sum_{k=1}^{n} \sum_{l=1}^{n-1} (-2l\phi' r_k^{(l)} + \tfrac{1}{2} l \phi^{(3)} r_k^{(l-1)}) \frac{\partial S_i}{\partial r_k^{(l)}}$$

$$+ \sum_{k=1}^{n} (-3\phi' s_k - 2\phi'' r_k^{(1)} + \tfrac{1}{2}\phi^{(4)} y_k) \frac{\partial S_i}{\partial s_k} \quad (i = 1, 2, \ldots, n-1);$$

whence
$$\frac{\delta S_1}{\delta t} = -[n(n-1)+1]\,\phi' S_1 - 2\phi'' R - \phi^{(3)} S_2,$$

$$\frac{\delta S_i}{\delta t} = -[n(n-1) - 2i + 3]\,\phi' S_i - \tfrac{1}{2}(i+1)\,\phi^{(3)} S_{i+1}$$

$$(i = 2, 3, \ldots, n-1),$$

where
$$S_n = 0.$$

Therefore S_{n-1} is a relative covariant. Also

$$\frac{\delta T_y}{\delta t} = -[n(n-1)+1]\,\phi' T_y + \tfrac{1}{2}\phi'' R - \tfrac{1}{2}\phi^{(3)} T_1,$$

and
$$\frac{\delta T_i}{\delta t} = -[n(n-1) - 2i + 1]\,\phi' T_i - \tfrac{1}{2}(i+1)\,\phi^{(3)} T_{i+1}$$

$$(i = 1, 2, \ldots, n-1),$$

where
$$T_n = 0.$$

Thus T_{n-1} is another relative covariant.

In forming the relative covariants we shall need before us the infinitesimal transformation of $I^{(r)}$ given by equation (12), viz.

$$\frac{\delta I^{(r)}}{\delta t} = -2(n-r)\,\phi' I^{(r)} + \tfrac{1}{2}(r+1)\,\phi^{(3)} I^{(r+1)}.$$

We can now write down the following series of relative covariants:

$$\Phi_n = R,$$

$$\Phi_{n-1} = S_{n-1},$$

$$\Phi_{n-2} = S_{n-2} + \frac{n-1}{n} I^{(n-1)} \Phi_{n-1},$$

$$\Phi_{n-3} = S_{n-3} + \frac{n-2}{n} I^{n-1} \Phi_{n-2} - \frac{(n-2)(n-1)}{n(n-1)} I^{(n-2)} \Phi_{n-1};$$

whence, by induction,

$$\Phi_{n-\alpha} = S_{n-\alpha} + \sum_{j=1}^{\alpha-1} (-1)^{j+1} \frac{(n-\alpha+j)!\,(n-j)!}{(n-\alpha)!\,n!} I^{(n-j)} \Phi_{n-\alpha+j}$$

$$(\alpha = 1, 2, \ldots, n-2).$$

Likewise

$$\Psi_{n-\alpha} = T_{n-\alpha} + \sum_{j=1}^{\alpha-1} (-1)^{j+1} \frac{(n-\alpha+j)!\,(n-j)!}{(n-\alpha)!\,n!} I^{(n-j)} \Psi_{n-\alpha+j}$$

$$(\alpha = 1, 2, \ldots, n-1).$$

Also $$\Phi_1 = 2\Theta_2 \left[S_1 + \sum_{j=1}^{n-2} (-1)^{j+1} \frac{(j+1)!\,(n-j)!}{n!} I^{(n-j)} \Phi_{j+1} \right] - R\Theta_2',$$

$$\Psi_0 = 8\Theta_2 \left[T_y + \sum_{j=1}^{n-1} (-1)^{j+1} \frac{j!\,(n-j)!}{n!} I^{(n-j)} \Psi_j \right] + R\Theta_2'.$$

If these $2n$ relative covariants are arranged in the order

$$\Phi_n, \Phi_{n-1}, \ldots, \Phi_1, \quad \Psi_{n-1}, \Psi_{n-2}, \ldots, \Psi_0,$$

they are seen to be independent, since each contains a semi-covariant not in those which precede. We therefore have all the independent covariants.

ON RESTRICTED FOURIER SERIES AND THE CONVERGENCE OF POWER SERIES

By Prof. W. H. Young.

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1. In a previous communication to the Society,* I introduced the term "restricted Fourier series of the first, second, . . . class," using the expression in the following sense :—

The p-th derived series of the Fourier series of $F(x)$ is said to be a restricted Fourier series of the p-th class, and to be restricted to one or more intervals (α, β), if, throughout each completely open interval in (α, β), $F(x)$ is a p-th integral.

The p-th differential coefficient of $F(x)$ in (α, β), defined almost everywhere, is then called the function associated with the restricted Fourier series.

The properties of a Fourier series retained by these trigonometrical series, in the interval or intervals to which they are restricted, were shewn in the memoir quoted to be, in general, what may be called "Cesàro-summation properties, index p." In particular, taking $p = 1$, the restricted Fourier series of the first class may be defined as trigonometrical series whose integrated series are Fourier series which converge in every completely open interval in (α, β) to a function $F(x)$ which is an absolutely convergent integral; and the properties of Fourier series retained by these trigonometrical series are all those involving Cesàro convergence, index 1. In particular the upper and lower functions, obtained by Cesàro summation $(C1)$, at a point x of the interval of restriction, are independent of the nature of the associated function, except in a neighbourhood of x as small as we please.

In a subsequent note,† presented to the Royal Society, I actually ob-

* "On the Convergence of the Derived Series of Fourier Series" (1916).

† "On the Ordinary Convergence of Restricted Fourier Series" (1917), *Proc. Roy. Soc.*, A, Vol. 93, pp. 276–292.

tained classes of restricted Fourier series which, without being themselves Fourier series, converged in the ordinary way throughout an interval. The method used was adapted *ad hoc;* I had, in fact, practical applications in view in the results I obtained. In the present communication my point of view is more theoretical and accordingly considerably wider in its consequences.

The point of departure is a remarkably simple theorem which I have been so fortunate as to discover. This theorem, while it more than justifies the introduction of the term "restricted Fourier series," and their separate consideration, seems to me, at the same time, to render desirable the distinction of a certain sub-class of restricted Fourier series as "ordinary restricted Fourier series," or, for brevity, "R.F. series."

Any trigonometrical series

$$\sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\},$$

—where for simplicity we have omitted the constant term—is defined to be an ordinary restricted Fourier series, or R.F. series, if it verifies two conditions:

(i) $$a_n \to 0, \quad b_n \to 0 \quad (n \to \infty);$$

(ii) *the integrated series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \{a_n \sin nx - b_n \cos nx\}$$

converges to an absolutely convergent integral

$$F(x) = \int f(x)\, dx,$$

in the interval or intervals of restriction $(\alpha < x < \beta)$.

The function associated with the R.F. series is then

$$f(x) = \frac{dF}{dx},$$

and is, of course, only defined almost everywhere in (α, β).

We may use the symbolic notation

$$f(x) \sim \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \quad (\alpha < x < \beta),$$

to denote that the series on the right is an R.F. series, with (α, β) as interval of restriction of $f(x)$ for associated function.*

The condition (i) is well-known to be verified by a Fourier series. *Thus R.F. series include Fourier series.* Also in virtue of (i), the squares of the coefficients of the integrated series form a convergent series, so that by the Riesz-Fischer theorem† the integrated series is a Fourier series. Thus *R.F. series are included in the class of restricted Fourier series of the first class.*

The object of the new term R.F. series is not, however, merely to connect the class with the sub-class, but to emphasize the extensive analogy between the new series and Fourier series, and thus at once to assist the memory and to render possible a generalisation of known theorems. It is evident that, the closer the analogy, the greater will be the advantage obtained by the nomenclature, and the less the danger of compensating disadvantages, due to the involuntary assumption that theorems true for Fourier series are also true for the extended class of ordinary restricted Fourier series (R.F. series), when this is not the case. *With the definition just given, almost all the more important properties of Fourier series are found to hold good for R.F. series in the interval or intervals to which we restrict ourselves.* Such a notable extension of our knowledge is rendered possible by the fundamental theorem above referred to.

I have, in fact, been able to prove that *the conditions for convergence and uniform convergence of an R.F. series in the interval of restriction are of precisely the same form as for a Fourier series. The same is true for the various possible forms of oscillation, uniform or otherwise.*

Again the same is true for the Cesàro convergence of the derived series of R.F. series : *the upper and lower functions of the p-th derived series of an R.F. series, when summed in the Cesàro manner (Cp), at any point internal to the interval of restriction, depend only on the nature of the associated function‡ in a neighbourhood of the point as small as we please.*

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* The symbol $\sim$ followed by $(-\pi < x < \pi)$ will then not be equivalent to the recognised symbol $\sim$ for a Fourier series, since the new symbolism does not even imply that the integrated series converges at the end-points $-\pi$ and π, or that, if it converges, the sum is other than discontinuous there, while $\sim$ implies both these facts, and also that $F(x)$ is an integral in the *closed* interval $(-\pi, \pi)$.

† For a collection of the existing proofs of this theorem see W. H. Young and Grace Chisholm Young, "On the Riesz-Fischer Theorem," *Quarterly Journal of Mathematics.*

‡ Which is, of course, the p-th differential coefficient of the function associated with the R.F. series from which our series is derived, since our series is a restricted Fourier series of the $(p+1)$-th class.

The condition imposed on the coefficients of the trigonometrical series considered—$a_n \to 0$, $b_n \to 0$ for an R.F. series, $n^{-p}a_n \to 0$, $n^{-p}b_n \to 0$ for its p-th derived series—is one which not only ensures what is desired, but is also necessary when the series is required to converge in the manner considered throughout an interval. I have not thought it necessary to insist on the somewhat more general theorems which arise when convergence at a point is required. In this case the condition may be relaxed,

$$(a_n \cos nx + b_n \sin nx) \to 0$$

in the former case, and

$$n^{-p}(a_n \cos nx + b_n \sin nx) \to 0$$

in the latter case, being then sufficient, as well as, of course, necessary.

It is not merely in the region of convergence that the analogy between Fourier series and R.F. series is complete. It is obviously also complete in the case, for example, of the chain of theorems, based on Riemann's work, due to Lebesgue, Fejér, de la Vallée Poussin, and myself, on certain conditions under which a trigonometrical series may be asserted to be a Fourier series. Nor do these theorems exhaust the list: only those in which the actual representation of the coefficients in terms of the associated function are required form exception.

The results to which we are led are not only important in the theory of trigonometrical series and its applications; they lead, when interpreted in the language of the complex variable, to a notable extension of our knowledge of the behaviour of power series on their circle of convergence. Our main theorem shews, for example, that *the uniform convergence of the power series* $\Sigma c_n z^n$, *where* $c_n \to 0$ $(n \to \infty)$, *can be asserted with confidence if* (i) *the sum-function has a unique limiting value at the point in question, and at the points of the circumference in its neighbourhood, and* (ii) *the function of* θ *formed by these limiting values on the circumference is a function of bounded variation.*

As before we may change the condition $c_n \to 0$ *into* $n^{-p}c_n \to 0$ $(n \to \infty)$, *provided we replace the ordinary convergence by Cesàro convergence* (Cp).

This theorem includes, as a very special corollary, Fatou's convergence theorem, the recognised importance of which in the Theory of Functions of a Complex Variable arises from the very fact that nothing is required to be known about the sum function, except in the neighbourhood of the point considered (the regularity, in fact, of the sum-function there): the only condition added is the necessary one $(c_n \to 0)$ for convergence.

The three proofs that have been given of Fatou's convergence theorem are sufficiently long and complicated; even the shortest, given by Marcel

Riesz, is considerably more difficult than our proof of the more general theorem, based as it is only on the essential features of the problem, and accordingly almost intuitive.

In this connexion it may not be out of place to remark that inadequate use appears still to have been made hitherto of real function theory in dealing with the complex variable, owing partly perhaps to the mistrust of the older analysts in the theory of the real variable, as concerned with the pathology rather than the morphology of functions, and this though it has given to mathematics in the new concept of integration perhaps the greatest impetus since the time of Riemann.

2. The main theorem is as follows :—

Theorem 1.—*The upper and lower functions of an R.F. series at a point x internal to its interval of restriction (α, β), depend only on the properties of the associated function in a neighbourhood of x as small as we please.*

Let the R.F. series be

$$f(x) \sim \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \quad (\alpha < x < \beta), \tag{1}$$

where, by definition, $$a_n \to 0, \quad b_n \to 0 \quad (n \to \infty). \tag{2}$$

The first integrated series will then be a Fourier series, say

$$F(x) \sim \sum_{n=1}^{\infty} \left\{ \frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right\}, \tag{3}$$

where, in the interval of restriction,

$$F(x) - F(\alpha) = \int_{\alpha}^{x} f(x)\, dx \quad (\alpha < x < \beta). \tag{4}$$

The sum s_n of n terms of the series (1) is then obtained by differentiating the sum of n terms of the series (3). Therefore

$$\begin{aligned} s_n = & \frac{1}{2\pi} \frac{d}{dx} \int_0^{\pi} \{F(x+u) + F(x-u)\} \sin(n+\tfrac{1}{2})u \operatorname{cosec} \tfrac{1}{2}u\, du \\ = & \frac{1}{2\pi} \frac{d}{dx} \int_0^{\epsilon} \{F(x+u) + F(x-u)\} \sin(n+\tfrac{1}{2})u \operatorname{cosec} \tfrac{1}{2}u\, du \\ & + \frac{1}{2\pi} \frac{d}{dx} \int_{\epsilon}^{\pi} \{F(x+u) + F(x-u)\} \sin(n+\tfrac{1}{2})u \operatorname{cosec} \tfrac{1}{2}u\, du, \end{aligned} \tag{5}$$

where e is chosen as small as we please, and certainly small enough for the interval $(x-e, x+e)$ to lie inside (α, β). The first of the two terms on the right in (5) depends only on the properties of $f(x)$ in a neighbourhood of x as small as we please. Thus our theorem only requires us to prove the second term in (5) to tend towards zero when $n \to \infty$. We may write the second term in the form

$$\frac{1}{2\pi}\frac{d}{dx}\int_e^\pi \{F(x+u)+F(x-u)\}\sin nu \cot \tfrac{1}{2}u\,du + \frac{1}{2\pi}\gamma_n, \tag{6}$$

where $$\gamma_n = \gamma_{-n} = \frac{d}{dx}\int_e^\pi \{F(x+u)+F(x-u)\}\cos nu\,du, \tag{7}$$

and our theorem will be proved if we shew that the expression (6) tends to zero when $n \to \infty$.

3. Now, since (3) is a Fourier series,

$$\frac{d}{dx}\int_0^\pi \{F(x+u)+F(x-u)\}\cos nu\,du = \pi\{a_n \cos nx + b_n \sin nx) \to 0 \quad (n \to \infty), \tag{9}$$

by (2).

Also since, in (α, β), $F(x)$ is the integral of $f(x)$,

$$\frac{d}{dx}\int_0^e \{F(x+u)+F(x-u)\}\cos nu\,du = \int_0^e \{f(x+u)+f(x-u)\}\cos nu\,du \to 0 \quad (n \to \infty), \tag{10}$$

by the theorem of Riemann-Lebesgue.*

From (9) and (10), by subtraction,

$$\gamma_n \to 0 \quad (n \to \infty). \tag{11}$$

Thus the condition to be proved becomes

$$Q_n = \frac{d}{dx}\int_e^\pi \{F(x+u)+F(x-u)\}\sin nu \cot \tfrac{1}{2}u\,du \to 0 \quad (n \to \infty). \tag{6a}$$

4. Now let $\phi(u)$ denote an odd function having the following properties:—

(i) $\phi(u) = \cot \frac{1}{2}u$ in (e, π), and in $(-e, -\pi)$;

(ii) $\phi(0) = 0$;

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* B. Riemann, *Habilitationschrift*, § 10 (1854); H. Lebesgue, *Leçons sur les séries trigonométriques* (1906), p. 61.

(iii) $\phi(u)$ is continuous at $u = e$ and $u = -e$;

(iv) $\phi(u)$ possesses its two* first differential coefficients at every point of the closed interval $(-e, e)$, and these are bounded.

This is clearly feasible, and the function $\phi(u)$ is then the integral of an integral. We have, therefore,

$$\phi(u) = \sum_{p=1}^{\infty} c_p \sin pu, \tag{1a}$$

where the series converges uniformly. The first differentiated series also converges uniformly and its sum is $\phi'(u)$.

The coefficients c_p then have the property

$$c_p = O(p^{-3}), \tag{13}$$

and the series formed by them is absolutely convergent.

Let us write
$$C = \sum_{1}^{\infty} |c_p|, \tag{14}$$

and
$$K_n = \sum_{n}^{\infty} |c_p|, \tag{15}$$

so that
$$K_n \to 0 \quad (n \to \infty). \tag{16}$$

5. Let us write also

$$G(x) - G(-\pi) = \int_{-\pi}^{x} F(x)\,dx \quad (-\pi \leqslant x \leqslant \pi).$$

We then have, integrating by parts,

$$\begin{aligned}\int_e^{\pi} \{F(x+u)+F(x-u)\} \sin nu\, \phi(u)\,du &= -\{G(x+e)-G(x-e)\} \sin ne\, \phi(e) \\ &\quad -n\int_e^{\pi} \{G(x+u)-G(x-u)\}\, \phi(u) \cos nu\,du \\ &\quad -\int_e^{\pi} \{G(x+u)-G(x-u)\}\, \phi'(u) \sin nu\,du. \end{aligned} \tag{17}$$

Now F is the differential coefficient of G at every point of the closed

* We can easily make as many of the differential coefficients exist as we please, or all of them: this is not required here, but will be required in § 9.

interval $(x-e,\ x+e)$ inside the interval of restriction $(\alpha,\ \beta)$, in which F, being an integral, is continuous: and, in $(e,\ \pi)$, F is almost everywhere equal to dG/dx. Therefore, differentiating (17), we get

$$Q_n = -\{F(x+e)-F(x-e)\}\sin ne\,\phi(e) - n\int_e^\pi \{F(x+u)-F(x-u)\}\cos nu\,du$$

$$-\int_e^\pi \{F(x+u)-F(x-u)\}\,\phi'(u)\sin nu\,du. \tag{18}$$

But since the Fourier series of ϕ and ϕ' converge uniformly, we may replace these functions by their Fourier series and integrate term-by-term. Having done this, we have only to transform back the expression under the sign of integration, $c_p \sin pu$ replacing $\phi(u)$, and we get

$$Q_n = \sum_{p=1}^{\infty} c_p \frac{d}{dx}\int_e^\pi \{F(x+u)-F(x-u)\}\sin nu \sin pu\,du$$

$$= \sum_{p=1}^{\infty} c_p \{\gamma_{p-n}-\gamma_{p+n}\}, \tag{19}$$

where γ_n is given by (7) and satisfies (11).

6. Therefore whatever positive quantity ϵ we choose, we can find a corresponding integer m such that, for all values of $r \geqslant m$,

$$|\gamma_r| < \epsilon. \tag{20}$$

Hence, when $n \geqslant m$,

$$\left|\sum_{p=1}^{\infty} c_p\gamma_{p+n}\right| \leqslant \epsilon \sum_{p=1}^{\infty}|c_p| \leqslant \epsilon C, \tag{21}$$

$$\left|\sum_{p=n+m+1}^{\infty} c_p\gamma_{p-n}\right| \leqslant \epsilon \sum_{p=n+m+1}^{\infty}|c_p| \leqslant \epsilon K_n \to 0 \quad (n\to\infty). \tag{22}$$

Writing

$$\sum_{p=1}^{n+m} c_p\gamma_{p-n} = \sum_{p=1}^{n-m} c_p\gamma_{n-p} + \sum_{p=n-m+1}^{n} c_p\gamma_{n-p} + \sum_{p=n+1}^{n+m} c_p\gamma_{p-n}$$

$$= \quad S_1 \quad + \quad S_2 \quad + \quad S_3,$$

we have

$$|S_1| = \left|\sum_{p=1}^{n-m} c_p\gamma_{n-p}\right| = \left|\sum_{r=m}^{n-1} c_{n-r}\gamma_r\right| \leqslant \epsilon \sum_{p=1}^{n-m}|c_p| \leqslant \epsilon C. \tag{23}$$

Also, if M be any positive quantity, not less than the absolute values

of the $m+1$ quantities $\gamma_0, \gamma_1, \ldots, \gamma_m$, we have

$$|S_2| = \left| \sum_{p=n-m+1}^{n} c_p \gamma_{n-p} \right| = \left| \sum_{r=0}^{m-1} c_{n-r} \gamma_r \right| \leqslant M \sum_{p=m-n+1}^{n} |c_p| \leqslant MK_{n-m} \to \infty \quad (n \to \infty), \quad (24)$$

and

$$|S_3| = \left| \sum_{p=n+1}^{n+m} c_p \gamma_{p-n} \right| = \left| \sum_{r=1}^{m} c_{n+r} \gamma_r \right| \leqslant M \sum_{p=n+1}^{n+m} |c_p| \leqslant MK_n \to 0, \quad (n \to \infty). \quad (25)$$

Using (21)–(25) in (19), we get

$$\operatorname*{Llt}_{n \to \infty} |Q_n| \leqslant 2\epsilon C,$$

whence, since ϵ is as small as we please,

$$Q_n \to 0 \quad (n \to \infty). \quad (6a)$$

Thus the required condition is satisfied, which proves the theorem.

7. In the preceding proof we have only used the fact that

$$a_n \cos nx + b_n \sin nx \to 0 \quad (n \to \infty),$$

namely in (9), and not the whole assumption $a_n \to 0$, $b_n \to 0$.

If, however, we require convergence in a whole interval, whether (α, β), or a part of (α, β), the coefficients a_n and b_n themselves must, as Riemann already remarked,* tend towards zero. *It is then easily seen that* $Q_n \to 0$ *uniformly.*

Indeed, we can find m *independent of* x, so that for $n \geqslant m$,

$$\left| \frac{1}{\pi} \frac{d}{dx} \int_0^{\pi} \{F(x+u) + F(x-u)\} \cos nu \, du \right| \leqslant |a_n| + |b_n| \leqslant \tfrac{1}{2}\epsilon. \quad (9a)$$

The uniformity of approach to zero of the corresponding integral whose upper limit of integration is e instead of π is part of the theorem of Riemann-Lebesgue. Hence

$$|\gamma_n| \leqslant \epsilon \quad (n \geqslant m), \quad (11a)$$

m *being independent of* x.

In § 6 the quantity M may be likewise chosen independent of x. In-

* *Loc. cit.*, § 7.

deed, as in § 3, introducing the argument x explicitly,

$$|\gamma_n(x)| = \left| \pi\{a_n \cos nx + b_n \sin nx\} - \int_0^e \{f(x+u)+f(x-u)\} \cos nu\, du \right|$$

$$\leqslant \pi\{|a_n|+|b_n|\} + \int_{x-e}^{x+e} |f(t)|\, dt.$$

Putting n in turn equal to 0, 1, ..., m, we see that we may put

$$M = \pi \sum_0^m \{|a_n|+|b_n|\} + m\int_\alpha^\beta |f(t)|\, dt,$$

and then M is, as it should be, not less than $\gamma_0(x)$, $\gamma_1(x)$, ..., $\gamma_m(x)$, wherever x may be in (α, β).

By (19) and (21)–(25) we have therefore, *m and M being independent of x,*

$$|Q_n(x)| \leqslant 2\epsilon C + \epsilon K_n + MK_{n-m} + MK_n \quad (n \geqslant m),$$

which shews that $Q_n(x)$ *converges uniformly to zero*, and proves our required result.

8. As an immediate corollary to Theorem 1, in conjunction with § 7, we have the following :—

THEOREM 2.—*If* $f(x) \sim \Sigma\{a_n \cos nx + b_n \sin nx\}$ $(\alpha < x < \beta)$, (1)

and $f_1(u)$ is any absolutely integrable function, agreeing with $f(u)$ in $(x-e, x+e)$, inside (α, β), then the R.F. series* (1) *and the Fourier series of $f_1(u)$ have the same peculiarities with regard to convergence or oscillation, whether at the point x or in the interval $(x-e, x+e)$.*

Also in all integration theorems involving term-by-term integration of a Fourier series, when multiplied by an absolutely integrable function $g(x)$, we may substitute an R.F. series for the Fourier series, provided the range of integration lie within the corresponding interval of restriction.

Hence we have for R.F. series the same criteria of convergence as for Fourier series. In particular Dirichlet's criterion may be applied, and we have the following theorem :—

THEOREM 3.—*The R.F. series of a continuous function of bounded variation converges uniformly to that function.*

9. As the first class of restricted Fourier series contains R.F. series,

* Summable.

the second class will contain the derived series of R.F. series. These will evidently be characterised among the series of the second class by the condition

$$n^{-1}a_n \to 0, \quad n^{-1}b_n \to 0 \quad (n \to \infty).$$

Generally the p-th derived series of R.F. series are characterised among the restricted Fourier series of the $(p+1)$-th class by the condition

$$n^{-p}a_n \to 0, \quad n^{-p}b_n \to 0 \quad (n \to \infty).$$

In a precisely similar manner to that used in proving our main theorem, we can prove the following more general theorem applying to these last-mentioned series:—

Theorem 4.—*The Cesàro upper and lower functions (index p) of the p-th derived series of an R.F. series, at a point x internal to its interval of restriction (α, β), depend only on the properties of the associated function in a neighbourhood of x as small as we please.*

It will be sufficient to remark that, in the case when $p = 1$, the discussion turns on the fact that, since $n^{-1}a_n \to 0$, $n^{-1}b_n \to 0$,

$$\frac{1}{2n\pi}\frac{d^2}{dx^2}\int_\epsilon^\pi \{G(x+u)+G(x-u)\}\sin^2\tfrac{1}{2}nt\,\mathrm{cosec}^2\tfrac{1}{2}t\,dt \to 0 \quad (n \to \infty)$$

—the approach to the limit being uniform in any interval internal to (α, β) —where $G(x) = \int_a^x dt \int f(t)dt$ is the function to which the second integrated series converges in (α, β).

This follows as before, bearing in mind that, when we replace $\sin^2\frac{1}{2}nt$ by $\frac{1}{2}(1-\cos nt)$, the part of our expression independent of $\cos nt$ has the limit zero as in § 3, the theorem of Riemann-Lebesgue applying, since we may now differentiate twice under the sign of integration. The remaining part of our expression is treated precisely as in §§ 4–6, the function $\phi(u)$ being provided with the proper array of differential coefficients.* The presence of the factor $1/n$ in our expression ensures the adequacy of the condition $n^{-1}a_n \to 0$, $n^{-1}b_n \to 0$.

10. Finally it may be remarked that there are corresponding theorems for fractional Cesàro convergence, in which the Fourier series of a function of bounded variation takes the place of that of an integral. Readers of my previous papers will be able to supply both the enunciations and the proofs of these results.

* See footnote, § 4.

In all cases the condition with respect to the coefficients of the trigonometrical series considered is not merely sufficient for its purpose, but necessary if convergence of the type considered in an interval is required. I have not thought it desirable to go at length into the more general theorems which arise when convergence at a point only is demanded: in this case the condition $n^{-p}a_n \to 0$, $n^{-p}b_n \to 0$ may be replaced by the less stringent requirement

$$n^{-p}\{a_n \cos nx + b_n \sin nx\} \to 0 \quad (n \to \infty).$$

11. Our present theory may be applied to the investigation of the behaviour of a power series on its circle of convergence, in the case when the coefficients tend towards zero. We shall assume the radius of convergence to be unity.

Let us denote the power series by

$$\sum_1^\infty c_n z^n = \sum_1^\infty (a_n - ib_n) z^n = \sum_1^\infty c_n r^n e^{in\theta}$$

$$= \sum_1^\infty r^n \{a_n \cos n\theta + b_n \sin n\theta\} - i \sum_1^\infty r^n \{b_n \cos n\theta - a_n \sin n\theta\},$$

where $$c_n \to 0 \quad (n \to \infty),$$

that is, $$a_n \to 0, \quad b_n \to 0 \quad (n \to \infty).$$

We have now to apply our theory simultaneously to a series

$$\sum_1^\infty r^n \{a_n \cos n\theta + b_n \sin n\theta\},$$

and to its allied series

$$\sum_1^\infty r^n \{b_n \cos n\theta - a_n \sin n\theta\}.$$

12. From Theorem 3 we have therefore at once the following :—

THEOREM 5.—*If a power series* $\sum_1^\infty c_n z^n$ *for which* $c_n \to 0$ $(n \to \infty)$, *is such that its second integrated series, which necessarily converges uniformly on the circle of convergence (of radius unity), has for sum a function of* θ *whose second differential coefficient is a continuous function of bounded variation* for all points of an arc, then the power series converges uniformly throughout that arc.*

* That is, $\phi_1(\theta) + i\phi_2(\theta)$, where ϕ_1 and ϕ_2 are continuous and of bounded variation.

13. Hence we get almost immediately the next theorem.

THEOREM.—*If the sum-function of a power series* $\sum_1^\infty c_n z^n$, *for which* $c_n \to 0$ $(n \to \infty)$, *has a unique limiting value* $f(\theta)$ *at each point* θ *of an arc* (α, β) *of the circle of convergence, and if* $f(\theta)$ *is then a function of bounded variation, the power series converges uniformly on that arc.*

For, if
$$f(r, \theta) = \sum_{n=1}^{\infty} c_n r^n e^{in\theta} \quad (r < 1),$$
then, by hypothesis,
$$\operatorname*{Lt}_{(r \to 1,\ \theta \to \theta_0)} f(r, \theta) = f(\theta_0) \quad (\alpha < \theta_0 < \beta).$$
Since this convergence to the limit $f(\theta_0)$ is uniform in (r, θ), $f(\theta_0)$ is continuous, and we may integrate twice with respect to θ, and get
$$\operatorname*{Lt}_{r \to 1} \int_\alpha^\theta d\theta \int_\alpha^\theta f(r, \theta)\, d\theta = \int_\alpha^\theta d\theta \int_\alpha^\theta f(\theta)\, d\theta,$$
that is,
$$\operatorname*{Lt}_{r \to 1} \left\{ -\sum_{n=1}^{\infty} n^{-2} c_n r^n e^{in\theta} \right\} = \int_\alpha^\theta d\theta \int_\alpha^\theta f(\theta)\, d\theta.$$
But since the series on the left converges uniformly, the sum is a continuous function of the ensemble (r, θ) ; this gives
$$-\sum_{n=1}^{\infty} n^{-2} c_n e^{in\theta} = \int_\alpha^\theta d\theta \int_\alpha^\theta f(\theta)\, d\theta,$$
whence, by the preceding theorem, the required result at once follows.

14. As a very special case of the preceding, we get Fatou's theorem :—

COR. (Fatou's Theorem).—*A power series* $\sum_{n=1}^\infty c_n z^n$, *for which* $c_n \to 0$ $n \to \infty$), *converges uniformly on any arc of the circle of convergence, provided that, at every point of the arc, the series is regular.*

In fact, if $\phi(z)$ be the analytic function represented by the power series $\sum_1^\infty c_n z^n$ inside the unit circle as circle of convergence, $\phi(z)$ is a continuous function of z with a continuous differential coefficient, not only inside the unit circle, but also on the given arc (α, β), since, by hypothesis, the series may be analytically continued over the arc (α, β). Thus, if, with the notation of the preceding article, we write
$$\phi(z) = f(r, \theta) \quad (r < 1),$$

and on the arc (α, β), $\phi(z_0) = f(\theta_0)$,

we have $$f(\theta_0) = \operatorname*{Lt}_{(r\to 1,\ \theta\to\theta_0)} f(r, \theta),$$

and $f(\theta)$ is an integral with respect to θ. Thus the conditions of the theorem are satisfied, whence the required result immediately follows.*

15. From Theorem 4 we get corresponding results in the theory of power series, the condition $c_n \to 0$ being replaced by $n^{-p}c_n \to 0$ $(n \to \infty)$, and ordinary convergence being changed into Cesàro convergence (Cp). Similarly, from § 10, we get further results of the same kind respecting the fractional Cesàro convergence of a power series.

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* Similar generalisations hold good for Marcel Riesz's extensions of Fatou's theorem.

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